Cost Sharing

Contents

1. Desirable Properties

Definition: (Immorlica et al., 2008) A cost-sharing method ξ is in the β-core w.r.t. costs C if ξ is β-BB and for all $S \subseteq T \subseteq [n]$ it holds that $\sum_{i \in S} \xi_i(T) \leq \beta \cdot C(S)$.

Note: Immediately by definition, any β-BB cross-montonic cost-sharing method is in the β-core.

Definition: (Deb and Razzolini, 1999) A cost-sharing mechanism M satisfies equal treatment if for all $i, j \in [n]$ and all $\boldsymbol{b} \in \mathbb{R}^n$ it holds that $b_i = b_j$ implies $M_i(\boldsymbol{b}) = M_j(\boldsymbol{b})$.

Definition: A social choice function f is pareto optimal if for every type vector $\boldsymbol{t}$ there is no outcome $o' \in \mathcal{O}$ so that $U(o') > U(f(\boldsymbol{t}))$.

Definition: (Penna et al.) A cost-sharing mechanism is renameproof if for all players $i, j \in [n]$, all true valuations $\boldsymbol{v}$ and all $\{i, j\}$-variants $\boldsymbol{b}$ with $b_j = v_i$ and $b_i = v_j = -1$ it holds that $u_j(\boldsymbol{b} \mid v_i) \leq u_i(\boldsymbol{v})$.

Definition: (Penna et al.) A cost-sharing mechanisms is reputationproof if the previous definition holds (at least) for all $j < i$.

1.1. Solution Concepts for Cooperative Games

Definition: Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_{\geq 0}$ be two allocations, $\sum x_i = \sum y_i$. Let $\widetilde{\cdot} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ sort the components of a vector. Vector $\boldsymbol{x}$ Lorenz dominates $\boldsymbol{y}$ iff for all $j \in [n]$ it holds that $\sum_{i=1}^j \widetilde{y}_i \leq \sum_{i=1}^j \widetilde{x}_i$, with at least one strict inequality.

Definition: (Dutta and Ray, 1989) The egalitarian solution is the set of all stable Lorenz-undominated allocations, where an allocation is stable if no (proper) subcoalition admits a better stable Lorenz-undominated allocation. (Better: At least one strictly improves, nobody becomes worse off.)

Formal definition: Given a cooperative game $v : 2^{[n]} \to \mathbb{R}_{\geq 0}$, define the Lorenz map and Lorenz cores

$$\begin{aligned} E^n A := \{\boldsymbol{x} \in A \mid \nexists \boldsymbol{y} \in A : \forall j \in [n] : & \\ & \textstyle\sum_{i=1}^j \widetilde{y}_i \leq \sum_{i=1}^j \widetilde{x}_i, \\ & \text{with at least one strict inequality}\} \\ L(S) := \{\boldsymbol{x} \in \mathbb{R}^S \mid \boldsymbol{x} \text{ feasible for } S \text{ and} & \\ & \nexists T \subsetneq S, \boldsymbol{y} \in E^{|T|} L(T) : \boldsymbol{y} > \boldsymbol{x}_T\}\,. \end{aligned}$$

The set of egalitarian allocations is $E^n L([n])$.

Theorem: There is at most one egalitarian allocation (called the egalitarian solution). It C is submodular, it can be computed be iteratively picking the most-cost efficient remaining set and assinging the respective prices. Otherwise, it may not exist.

E.g., if $n = 3$, $C(\{1, 2, 3\}) = 2$ and $C(S) = 1$ for all other non-empty S, then the egalitarian solution does not exist.

Note: The motivation of the egalitarian solution is to reconcile egalitarianism and stability.

Definition: Given a non-negative random variable T with distribution G and density g, the hazard rate $h : \mathbb{R}_{>0} \to \mathbb{R}$ is defined as

$$h(t) := \lim_{\Delta \to 0} \frac{\Pr[t \leq T \leq t + \Delta \mid t \leq T]}{\Delta} = \frac{g(t)}{1 - G(t)}\,.$$

Note: The hazard rate is the rate that an event occurs at t, given that it has not occurred before t.

Theorem: (Mutuswami, 2004) Suppose that all valuations v_i are independent draws from a common distribution function whose support is $(0, m)$. Suppose the hazard rate is non-decreasing, i.e., $h'(x) \geq 0$. Then, the egalitarian solution maximizes the probability that a given subset S of players can afford the cost shares $\xi(S)$.

Theorem: For submodular cost functions, the egalitarian solution is unique. The corresponding cost-sharing method is cross-monotonic.

Proof: Suppose $T = S \cup j$. Computing the egalitarian method partitions S into $S_1 \cup S_2 \cup \ldots$ and T into $T_1 \cup T_2 \cup \ldots$. Now, by way of contradiction, suppose there is

a k (w.l.o.g., this k is minimal) so that there is a player $i \in S_k$ with $\xi_i(S) > \xi_i(S \cup j)$. Let m be defined by $i \in T_m$.

Define $V_k := \bigcup_{j=1}^{k} S_j$ and $W_m := \bigcup_{j=1}^{m} T_j$. First note $V_{k-1} \subseteq W_{m-1}$. Otherwise, there is a $j \in V_{k-1}$ with $j \notin W_{m-1}$, so $\xi_j(S) \leq \xi_i(S) < \xi_i(T) \leq \xi_j(T)$. A contradiction to the choice of k as being minimal.

Define $Z := W_{m-1} \cap S_k$. By definition of T_m, submodularity, and inserting Z,

$$\begin{aligned}
\xi_i(T) &= \frac{C(W_{m-1} \cup T_m) - C(W_{m-1})}{|T_m|} \\
&\leq \frac{C(W_{m-1} \cup V_k) - C(W_{m-1})}{|V_k \setminus W_{m-1}|} \\
&\leq \frac{C(V_k) - C(W_{m-1} \cap V_k)}{|V_k \setminus W_{m-1}|} \\
&= \frac{C(V_k) - C(V_{k-1} \cup Z)}{|S_k \setminus Z|} \\
&= \frac{C(V_k) - C(V_{k-1}) - [C(V_{k-1} \cup Z) - C(V_{k-1})]}{|S_k| - |Z|} \\
&\leq \frac{C(V_k) - C(V_{k-1})}{|S_k|} = \xi_i(S)\,.
\end{aligned}$$

Here, the last inequality holds because

$$\frac{C(V_k) - C(V_{k-1})}{|S_k|} \leq \frac{C(V_{k-1} \cup Z) - C(V_{k-1})}{|Z|}$$

by definition of S_k and $\frac{a}{b} \leq \frac{c}{d} \Leftrightarrow ad \leq cb \Leftrightarrow ab - cb \leq ab - ad \Leftrightarrow \frac{a-c}{b-d} \leq \frac{a}{b}$. □

Definition: (See, e.g., Osborne and Rubinstein (1994, p. 290)) A value is a function that assigns a *unique* feasible payoff profile to every coalitional game with transferable payoff.

Definition: (Shapley, 1953) Given a cooperative game v and a set of players $S \subseteq [n]$, the Shapley-value contributions are defined as

$$\xi_i(S) := \sum_{T \subseteq S \setminus i} \frac{|T|! \cdot (|S| - |T| - 1)!}{|S|!} \cdot [v(T \cup i) - v(T)]\,.$$

Definition: (Hart and Mas-Colell, 1989) Given a game $v : 2^{[n]} \to \mathbb{R}$, a function $P : 2^{[n]} \to \mathbb{R}$ with $P(\emptyset) = 0$ is called a potential function if for all $S \subseteq [n]$

$$\sum_{i \in S} D^i P(S) = v(S)\,,$$

where $D^i P(S) := P(S) - P(S \setminus i)$.

Theorem: (Hart and Mas-Colell, 1989) There exists a unique potential function P, and $(D^i P(S))_{i \in S}$ coincides exactly with the Shapley value for coalition S.

Lemma: It holds that

$$P(S) = \sum_{T \subseteq S} \frac{(t-1)!(s-t)!}{s!} \cdot v(T)\,.$$

where $t = |T|$ and $s = |S|$.

Proof: Denote by $\xi : 2^{[n]} \to \mathbb{R}^n$ the Shapey-value contributions. Note that

$$P(S \setminus i) = \sum_{T \subseteq S \setminus i} \frac{(t-1)!(s-t-1)!}{(s-1)!} \cdot v(T)\,.$$

We have $P(S) - P(S \setminus i)$

$$\begin{aligned}
&= \sum_{T \subseteq S \setminus i} \frac{(t-1)!(s-t-1)!}{s!} \cdot ((s-t) - s) \cdot C(T) \\
&+ \sum_{T \subseteq S | T \ni i} \frac{(t-1)!(s-t)!}{s!} \cdot C(T) \\
&= - \sum_{T \subseteq S \setminus i} \frac{t! \cdot (s-t-1)!}{s!} \cdot C(T) \\
&+ \sum_{T \subseteq S \setminus i} \frac{t! \cdot (s-t-1)!}{s!} \cdot C(T \cup i)
\end{aligned}$$

By the previous characterization, this completes the proof. □

Note: If ξ contains the Shapley-value contributions, then for all orders $s_1, \ldots, s_{|S|}$ of S it holds that

$$\sum_{i=1}^{|S|} \xi_{s_i}(\{s_1, \ldots, s_i\}) = P(S)\,.$$

The Shapley value is the only value with this property.

The Shapley value can also be characterized in other ways. E.g., it is the only value that satisfies all of the following:

- feasible, $\sum \xi_i(S) = v(S)$
- anonymous
- additive
- $\xi_i(S) = 0$ if marginal costs of i are always 0

Theorem: (Sprumont, 1990) Given that C is submodular, the Shapley value is cross-monotonic.

Proof: The Shapley value is the average, over all orderings of the players, of the marginal cost distributions $\xi_i(S) = C(S \cap [i]) - C(S \cap [i-1])$. If C is submodular, then it is immediate that ξ is cross-monotonic.

Shapley (1971) proved that the core of a convex game is a polytope whose extreme points are the (usual) marginal contributions vectors. Now, a convex combination of cross-monotonic cost-sharing methods is clearly cross-monotonic, too. □

Theorem: Let $U \neq \emptyset$ be a finite set, $C : 2^U \to \mathbb{R}$ be a set function. The following two statements are equivalent:

i) for all $A, B \subseteq U : C(A) + C(B) \geq C(A \cup B) + C(A \cap B)$

ii) for all $D \subseteq E \subseteq U$ and $i \notin E : C(D \cup i) - C(D) \geq C(E \cup i) - C(E)$

Proof: For the proof, we rewrite (i). For all $A, B \subseteq U$: $C(A) - C(A \cap B) \geq C(A \cup B) - C(B)$.

"$\Rightarrow$": Let $D \subseteq E \subseteq U$ and $i \in U$. Setting $A := D \cup \{i\}$ and $B := E$ gives the desired result: $C(D \cup i) - C(D) \geq C(E \cup i) - C(E)$.

"$\Leftarrow$": Let $A, B \subseteq U$ and let $a_1, \ldots, a_n$ denote the elements of $A \setminus B$ (in any arbitrary order). By assumption, we have $C((A \cap B) \cup a_1) - C(A \cap B) \geq C(B \cup a_1) - C(B)$ and similarly

$$\begin{aligned} & C((A \cap B) \cup \{a_1, \ldots, a_k\}) \\ & - C((A \cap B) \cup \{a_1, \ldots, a_{k-1}\}) \\ \geq\ & C(B \cup \{a_1, \ldots, a_k\}) - C(B \cup \{a_1, \ldots, a_{k-1}\}) \end{aligned}$$

for all $k \in [n]$. Summing each side up for all $k \in [n]$ gives the desired result. □

2. Design Techniques

2.1. Mechanism Design Basics

Definition: Every player i is characterized by his type $t_i \in T_i$, which determines his preference over different outcomes. That is, $U_i(o \mid t_i)$ is the utility of player i with type t_i for outcome $o \in \mathcal{O}$.

Definition: A social choice function $f : T_1 \times \cdots \times T_n \to \mathcal{O}$ chooses an outcome $f(\boldsymbol{t}) \in \mathcal{O}$, given types $\boldsymbol{t} = (t_1, \ldots, t_n)$.

Definition: A social welfare function $F : T_1 \times \cdots \times T_n \times \mathcal{O} \to \mathbb{R}$ ranks conceivable social states.

Definition: A mechanism $g : S_1 \times \cdots \times S_n \to \mathcal{O}$ defines the set of strategies S_i available to each player i, and an outcome rule such that $g(\boldsymbol{s})$ is the outcome implemented by the mechanism for strategy profile $\boldsymbol{s} = (s_1, \ldots, s_n)$.

Theorem: (Revelation Principle) If there exists a mechanism g that implements a social choice function f in dominant strategies, then f is also truthfully implementable in dominant strategies (i.e., with a strategyproof mechanism).

2.2. Impossibility results

Definition: Player i's utilities are general when for every complete and transitive ordering $\succ$ of the outcome set $\mathcal{O}$ there is a type t_i so that $U_i(\cdot \mid t_i)$ induces $\succ$.

Definition: neutral, unanimity, irrelevant alternatives

Theorem: (Arrow, 1963) If players have general utilities, then every social welfare function over a set of more than 2 alternatives that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

Note: A well-known special case is the Condorcet paradox. Suppose $n = 3$ and $\mathcal{O} = \{A, B, C\}$ with $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, and $C \succ_3 A \succ_3 B$. By pairwise comparison, we get $A \succ B$, $B \succ C$, and $C \succ A$.

Theorem: (Gibbard, 1973; Satterthwaite, 1975) If players have general utilities and f is an incentive-compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.

2.3. Restricted Domains

Definition: Suppose $\mathcal{O} = A \times \mathbb{R}^n$, i.e., an outcome consists of an alternative $a \in A$ and monetary transfers $\boldsymbol{p} \in \mathbb{R}^n$. We decompose a social choice $f(\boldsymbol{t})$ into a choice rule $q(\boldsymbol{t})$ and a payment rule $x(\boldsymbol{t})$.

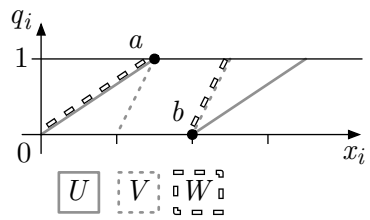
Definition: A choice rule q is implementable if there is a payment rule x so that (q, x) is implementable.

Definition: Player i's utilities are quasi-linear when the type of each player $i \in [n]$ is a valuation function $t_i : A \to \mathbb{R}$ so that $U_i(a, \boldsymbol{p} \mid t_i) = t_i(a) + p_i$.

In an unpublished paper, Shenker (1993) proved several results on the relationship between various forms of truthfulness, non-bossiness, and other technical properties. However, his results do not apply in settings with quasi-linear utilities.

Definition: A domain of utility functions $\mathcal{U}$ is monotonically closed if for all $U, V \in \mathcal{U}$ and all $a, b \in A$ with (1) $U(a) \geq U(b) \Rightarrow V(a) \geq V(b)$ and (2) $U(a) > U(b) \Rightarrow V(A) > V(b)$ there is a utility function $W \in \mathcal{U}$ so that for all $c \in A$ it holds that (3) $U(a) \geq U(c) \Rightarrow W(a) \geq W(c)$ and (4) $V(b) \geq V(c) \Rightarrow W(b) \geq W(c)$.

In general, the domain of quasi-linear utility functions is not monotonically closed:



A choice function $q : T \to A$ is called an affine maximizer if for some subrange $A' \subseteq A$, so some player weights $w_1, \ldots, w_n \in \mathbb{R}_{>0}$ and for some outcome weights $c_a \in R$, where $a \in A'$, we have that $f(\boldsymbol{t}) \in \arg\max_{a \in A'} \left(c_a + \sum_{i \in [n]} w_i \cdot t_i(a) \right)$.

Theorem: (Roberts, 1979) Suppose $|A| \geq 3$, $q : T_1 \times \cdots \times T_n \to a$ is a choice rule onto A, and $T_i = \mathbb{R}^A$ for all $i \in [n]$. Then q is implementable in dominant strategies (= truthfully implementable due to the revelation principle) if and only if q is an affine maximizer.

Definition: A choice rule q satisfies weak monotonicity (WMON) if for all players $i \in [n]$, and all i-variants $\boldsymbol{t}, \boldsymbol{t}'$ with $a := q(\boldsymbol{t}) \neq q(\boldsymbol{t}') =: b$ it holds that $t_i(a) - t_i(b) \geq t'_i(a) - t'_i(b)$.

Theorem: If a choice rule q is implementable in dominant strategies then q satisfies WMON. Conversely, if f satisfied WMON and all $T_i \subseteq \mathbb{R}^A$ are convex sets, then f is implementable in dominant strategies.

Note: It is known that WMON is not a sufficient condition for dominant strategy implementability.

Definition: A domain of quasi-linear utilities is called single-parameter when each valuation function is determined by a single real parameter.

Note, e.g., that much effort has been spent for devising monotone (in the machine speeds) approximation algorithms for makespan minimization on parallel related machines. A randomized PTAS is due to Dhangwatnotai et al. (2008).

2.4. Groves Mechanisms

Definition: (Groves, 1973) A cost-sharing mechanism $M = (Q, x)$ is a Groves mechanism with respect to costs C if for all $\boldsymbol{b} \in \mathbb{R}^n$ and players $i \in [n]$ it holds that

$$Q(\boldsymbol{b}) \in \arg\max_{T \subseteq [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\}$$
$$x_i(\boldsymbol{b}) = C(Q(\boldsymbol{b})) - \sum_{j \in Q(\boldsymbol{b}) \setminus i} b_j + h_i(\boldsymbol{b}_{-i}),$$

where $h_i : \mathbb{R}^{[n] \setminus i} \to \mathbb{R}$ is a function independent of b_i.

Note: Groves mechanisms are also called *VCG mechanisms* (Nisan, 2007, p. 218) or *Clarke-Groves* mechanisms (Moulin, 1999, p. 521). The name *Groves mechanisms* is used, e.g., in (Parkes, 2001, p. 41).

Theorem: Groves mechanisms are SP.

Proof: Assuming $h_i \equiv 0$, it holds for any i-variant $\boldsymbol{b}$ of $\boldsymbol{v}$ that

$$u_i(\boldsymbol{b}) = v_i \cdot q_i(\boldsymbol{b}) + \sum_{j \in Q(\boldsymbol{b}) \setminus i} b_j - C(Q(\boldsymbol{b}))$$
$$= \sum_{j \in Q(\boldsymbol{b})} v_j - C(Q(\boldsymbol{b})) \le u_i(\boldsymbol{v}),$$

where the last inequality holds because $Q(\boldsymbol{b})$ can, by definition, only be inferior to $Q(\boldsymbol{v})$. □

Theorem: Any cost-sharing mechanism that is both SP and 1-EFF is a Groves mechanism.

Note: For the domain with arbitrary valuation functions, a corresponding statement has been proven by Green and Laffont (1977). For the (single-parameter) cost-sharing model, the proof is easier.

Proof: Note first that a mechanism $M = (Q, x)$ is a Groves mechanism if and only if for all i-variants $\boldsymbol{b}, \boldsymbol{b}'$ it holds that

i) $Q(\boldsymbol{b}) \in \arg\max_{T \in [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\}$,

ii) when $S := Q(\boldsymbol{b})$, $T := Q(\boldsymbol{b}')$ then $x_i(\boldsymbol{b}) - x_i(\boldsymbol{b}') = \left[C(S) - \sum_{j \in S \setminus i} b_j \right] - \left[C(T) - \sum_{j \in T \setminus i} b_j \right]$.

By way of contradiction, suppose now that (ii) does not hold for some pair $\boldsymbol{b}, \boldsymbol{b}'$. Denote by $\boldsymbol{s}, \boldsymbol{t}$ the respective allocations. If $s_i = t_i$ then $x_i(\boldsymbol{b}) = x_i(\boldsymbol{b}')$ due to the threshold property. Moreover, $C(S) = C(T)$ due to (i), so also $\left[C(S) - \sum_{j \in S \setminus i} b_j \right] - \left[C(T) - \sum_{j \in T \setminus i} b_j \right]$.

W.l.o.g. consider the case $i \in S$ but $i \notin T$. Due to the threshold property, we have

$$\sum_{j \in S \setminus i} b_i + \theta_i(\boldsymbol{b}_{-i}) - C(S) = \sum_{j \in T \setminus i} b_i - C(T),$$

i.e., $x_i(\boldsymbol{b}) - x_i(\boldsymbol{b}') = \theta_i(\boldsymbol{b}_{-i}) = \left[C(S) - \sum_{j \in S \setminus i} b_j \right] - \left[C(T) - \sum_{j \in T \setminus i} b_j \right]$. □

Definition: (Moulin and Shenker, 2001) A cost-sharing mechanism is a marginal cost pricing mechanism if it is a Groves mechanism with

$$Q(\boldsymbol{b}) = \max \left(\arg\max_{T \subseteq [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\} \right)$$
$$h_i(\boldsymbol{b}_{-i}) = \max_{T \subseteq [n] \setminus i} \left\{ \sum_{j \in T} b_j - C(T) \right\}.$$

Here, the max around the arg max chooses the maximum subset, which is well-defined.

Theorem: (Moulin and Shenker, 2001) Any cost-sharing mechanism M (satisfying NPT and VP) that is both SP and 1-EFF if and only if it welfare-equivalent to a marginal cost pricing mechanism.

Proof: "$\Leftarrow$": Due to

$$\sum_{j \in Q(\boldsymbol{b}) \setminus i} b_j - C(Q(\boldsymbol{b})) \le h_i(\boldsymbol{b}_{-i}) \le \sum_{j \in Q(\boldsymbol{b})} b_j - C(Q(\boldsymbol{b}))$$

we have $x_i(\boldsymbol{b}) \in [0, b_i \cdot q_i(\boldsymbol{b})]$, i.e., NPT and VP hold.

"$\Rightarrow$": From $b_i = 0$ we have $h_i(\boldsymbol{b}_{-i}) = \sum_{j \in Q(\boldsymbol{b}) \setminus i} b_j - C(Q(\boldsymbol{b}_{-i}, 0))$, which is the same as for the marginal cost pricing mechanism.

Now suppose i is served by the marginal cost pricing mechanism but not by M (all other players receive precisely the same outcome). Then $x_i(\boldsymbol{b}) = b_i$, which proves the claim. □

2.5. Primal-Dual Algorithms

The primal-dual schema for solving IPs:

i) Write down LP relaxation, find dual. Find some intuitive meaning for the dual variables.

ii) Start with vectors $\boldsymbol{x} = 0$, $\boldsymbol{y} = 0$, which are dual feasible, but primal infeasible.

iii) Until the primal is feasible

   (a) increase the dual values y_i until some dual contraint(s) go(es) tight, while maintaining dual feasibility

(b) Select some subset of the tight dual contraints, and increase the primal variable corresponding to them by an integral amount

iv) For the analysis, prove that the output pair of vectors satisfies $\sum_i c_i x_i \leq \rho \cdot \sum b_j y_j$ for as small a value of ρ as possible.

Beispiel: Facility Location: Input is bipartite graph $G = (V, E)$, where $V = F \cup N$. Each facility $v \in F$ has opening costs f_v, and each edge $e \in E$ is associated with a connection cost c_e. The triangle inequality is fulfilled. A star $S = (f, N')$ consists of a facility f and a set of connected users N'. The set of all starts is denoted by $\mathscr{S}$. The cost c_S for a star $S = (v, N')$ is $c_S = f_v + \sum_{i \in N'} c_{vi}$.

Formulation as IP:

$$\begin{aligned} \min \quad & \sum_{S \in \mathscr{S}} c_S \cdot y_S \\ \text{s.t.} \quad & \sum_{S \ni i} y_S \geq 1 && \text{for all } i \in C \\ & y_S \in \{0, 1\} && \text{for all } S \in \mathscr{S} \end{aligned}$$

Dual for LP relaxation:

$$\begin{aligned} \max \quad & \sum_{i \in N} \xi_i \\ \text{s.t.} \quad & \sum_{i \in S} \xi_i \leq c_S && \text{for all } S \in \mathscr{S} \\ & \xi_i \geq 0 && \text{for all } i \in N \end{aligned}$$

The primal-dual algorithm where dual variables is frozen once the corresponding stars are "tight" is the the basis of the SP cost-sharing mechanism by Devanur et al. (2005). Their mechanism can as well be interpreted as an egalitarian mechanisms; the next set that "goes tight" is clearly the "most cost-effective" set.

A cross-monotonic cost-sharing method is obtained when cost shares correspond to the dual variables at the time they become tight; however, the dual variables continue to be increased (Pál and Tardos, 2003).

2.6. Cost-Sharing Mechanisms

Acyclic mechanisms: For every subset of players $S \subseteq [n]$, consider the following directed graph. The vertices are the players in S and there is an edge (i, j) iff $\xi_j(S \setminus i) < \xi_j(S)$, i.e., player j benefits from removing player i. This graph is acyclic. In particular, if there is an edge (i, j), then the offer time $\tau_j(S) > \tau_i(S)$.

2.7. Lower Bounds for Cross-Monotonicity

Many lower bounds on the approximate budget balance of families of cost-sharing mechanisms have been shown. Immorlica et al. (2008) were the first to systematically establish such bounds.

In the edge-cover cost-sharing problem, we are given a graph $G = (V, E)$ without isolated vertices. The set of players is V, and $C(S) =$ min. size of a set of edges that spans all vertices in S.

Theorem: (Immorlica et al., 2008) In general, cross-monotonic cost-sharing methods for the edge cover problem are no better than 2-BB.

Proof: (using the probabilistic method) Let $K_{n,n}$ denote the complete bipartite graph with $2n$ nodes. For $v \in V$, denote by S_v the union of v with its adjacent nodes (i.e., with the other partition). Clearly, $C(S_v) = n$.

Let ξ be a β-BB cost-sharing method, and consider now an arbitrary node $v \in V$ picked uniformly at random.

$$\begin{aligned} \mathrm{E}_v\left[\sum_{i \in S_v} \xi_i(S_v)\right] &\leq \mathrm{E}_v\left[\xi_v(\{v\})\right] + \mathrm{E}_v\left[\sum_{u \in S_v \setminus v} \xi_u(\{u, v\})\right] \\ &\leq \beta + n \cdot \mathrm{E}_{u|v}\left[\xi_u(\{u, v\})\right] \\ &= \beta + n \cdot \mathrm{E}_{\{u,v\}}\left[\xi(\{u, v\})\right] \\ &\leq \beta + n \cdot \frac{\beta}{2} = \beta \cdot \left(1 + \frac{n}{2}\right). \end{aligned}$$

It follows that there is a $\boldsymbol{v} \in V$ so that

$$1 \leq \frac{\sum_{i \in S_v} \xi_i(S_v)}{C(S_v)} \leq \frac{\beta \cdot \left(1 + \frac{n}{2}\right)}{n} = \beta \cdot \left(\frac{1}{n} + \frac{1}{2}\right)$$

where the first inequality is due to β-BB (cost recovery). Consequently, $\beta \geq \frac{2n}{n+2}$. □

Theorem: In general, coss-monotonic cost-sharing methods for the makespan problem with identical machines and identical jobs are no better than 2-BB.

Proof: Let ξ be a β-BB cost-sharing method. Suppose there are $m = n - 1$ machines, i.e., $C(S) = 1$ for $S \subsetneq [n]$ and $C([n]) = 2$. Hence, there is a player i with $\xi_i([n]) \leq \frac{\beta}{m}$. Then,

$$\begin{aligned} 2 \leq \sum_{j \in [n]} \xi_j([n]) &\leq \frac{\beta}{m} + \sum_{j \in [n] \setminus i} \xi_i([n]) \leq \frac{\beta}{m} + \beta \\ &= \frac{m+1}{m} \cdot \beta, \end{aligned}$$

i.e., $\beta \geq \frac{2m}{m+1}$. □

Definition: (Roughgarden and Sundararajan, 2006) A cost-sharing method ξ is α-summable with regard to costs C if for all sets $S \subseteq [n]$ and all orders $s_1, \ldots, s_{|S|}$ of S it holds that

$$\sum_{i=1}^{|S|} \xi_{s_i}(\{s_1, \ldots, s_i\}) \leq \alpha \cdot C(S).$$

Theorem: (Roughgarden and Sundararajan, 2006) Suppose ξ is a cross-monotonic cost-sharing method that is β-BB with regard to approximate costs C' and optimal costs C, and moreover α-summable with regard to C. Then, the Moulin mechanism induced by ξ is α-EFF.

Proof: Let Q be the set returned by the Moulin mechanism for $\boldsymbol{v}$, and let P be a set with optimum social cost. For the analysis, renumber the player so that $P \setminus Q = \{m+1 \dots n\}$, and the players in $P \setminus Q$ are deleted by the Moulin mechanism in *descending* order of their numbers. For convenience, define $P_i := P \cap [i]$. Then,

$$\begin{aligned} SC(Q) &= C'(Q) + \sum_{i \notin Q} v_i \leq \sum_{i \in Q} \xi_i(Q) + \sum_{i \notin Q} v_i && (1) \\ &\leq \sum_{i \in Q \cap P} \xi_i(Q) + \sum_{i \notin Q \cap P} v_i && (2) \\ &< \sum_{i \in Q \cap P} \xi_i(P_i) + \sum_{i \in P \setminus Q} \xi_i(P_i) + \sum_{i \notin P} v_i && (3) \\ &\leq \alpha \cdot C(P) + \sum_{i \notin P} v_i \,. \end{aligned}$$

Note that (1) is due to BB and (2) is due to $\xi_i(Q) \leq v_i$ for $i \in Q$. Finally, (3) holds because $P_i \subseteq Q$ for $i \in Q \cap P$ (as $P \setminus Q = \{m+1 \dots n\}$) and because when $i \in P \setminus Q$ was dropped, all players in P_i were still in the game. □

2.8. General Lower Bounds

Theorem: (Feigenbaum et al., 2003) In general, no SP cost-sharing mechanism can guarantee both approximate budget-balance and an approximation of the social welfare.

Proof: Consider the excludable public case, $C(S) = 1$ for $S \neq \emptyset$. Let $v_i = \frac{1}{n-1}$. Then, $Q(\boldsymbol{v}) = [n]$ because otherwise the social welfare is 0 whereas $\sum_{i \in [n]} v_i - 1 = \frac{1}{n-1}$. Now, for every $\varepsilon > 0$ we have $i \in Q(\boldsymbol{v}_{-i}, \varepsilon)$ due to the approximation guarantee of the social welfare. Consequently, $x_i(\boldsymbol{v}) = \theta_i(\boldsymbol{v}_{-i}) = 0$, and the mechanism is not approximate budget-balanced. □

Theorem: (Dobzinski et al., 2008) Any SP and β-BB cost-sharing mechanism $M = (Q, x)$ is $\Omega(\log n)$-EFF.

Proof: Probabilistic method: There is a "bad" instance. Let $\lambda \geq 1$ be a parameter and suppose the true valuations of each player are i.i.d. random variables with $\Pr[v_i = 0] = \frac{1}{n}$ and for $1 \leq t \leq n$:

$$\Pr\left[v_i \leq \frac{t}{\lambda n}\right] = \frac{1}{n} + \int_1^t \frac{1}{z^2}\, dz = \frac{1}{n} + 1 - \frac{1}{t}$$

Therefore,

$$\begin{aligned} \mathrm{E}\left[v_i\right] &= \int_{1/\lambda n}^{1/\lambda} z \cdot \frac{1}{\lambda n z^2}\, dz = \frac{1}{\lambda n} \cdot \left[\ln n\right]_{1/\lambda n}^{1/\lambda} \\ &= \frac{-\ln \lambda + \ln(\lambda n)}{\lambda n} = \frac{\ln n}{\lambda n} \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\left[v_i^2\right] &= \int_{1/\lambda n}^{1/\lambda} z^2 \cdot \frac{1}{\lambda n z^2}\, dz = \frac{1}{\lambda n} \cdot \left(\frac{1}{\lambda} - \frac{1}{\lambda n}\right) \\ &= \frac{n-1}{\lambda^2 n^2} \end{aligned}$$

Define $V = \sum v_i$. Then, $\mathrm{E}[V] = \frac{\ln n}{\lambda}$ and $\mathrm{Var}[V] = \sum \mathrm{Var}[v_i] \leq n \cdot \mathrm{E}[v_i^2] = \frac{1}{\lambda^2}$. Hence, Chebyshev's inequality yields for $t > 0$ that

$$\begin{aligned} \Pr\left[\frac{\ln n}{\lambda} - t \leq V\right] &\geq \Pr\left[\left|V - \frac{\ln n}{\lambda}\right| \leq t\right] \\ &= 1 - \Pr\left[\left|V - \frac{\ln n}{\lambda}\right| \geq t\right] \\ &\geq 1 - \frac{\mathrm{Var}[V]}{t^2} \\ &= 1 - \frac{1}{\lambda^2 t^2} \end{aligned}$$

Let $\theta := \theta_i(\boldsymbol{v}_{-i})$. Then, the expected revenue from player i is

$$\theta \cdot \Pr[v_i > \theta] = \theta \cdot \left(\frac{1}{\theta \lambda n} - \frac{1}{n}\right) = \frac{1}{\lambda n} - \frac{\theta}{n} \leq \frac{1}{\lambda n}\,.$$

Consequently, the total expected revenue is at most $\frac{1}{\lambda}$. Then, since M is β-BB, we have $\Pr[Q(\boldsymbol{v}) = \emptyset] \geq 1 - \frac{1}{\lambda}$. Now,

$$\Pr\left[\ln n - t \leq V \text{ and } Q(\boldsymbol{v}) = \emptyset\right] \geq 1 - \frac{1}{\lambda^2 t^2} - \frac{1}{\lambda}\,.$$

by the union bound. Hence, the social cost is at least $\frac{\ln n}{\lambda} - t$ with strictly positive probability. However, the optimal social cost is 1. □

Theorem: (Moulin and Shenker, 2001; Roughgarden and Sundararajan, 2006) Given submodular costs C, Moulin mechanisms driven by the Shapley value are H_n-EFF.

Proof: It suffices to show H_n-summability. Suppose $s_1, \dots, s_{|S|}$ is an arbitrary order of S. It holds that

$$\begin{aligned} \sum_{i=1}^{|S|} \xi_{s_i}(\{s_1, \dots, s_i\}) = P(S) &= \sum_{T \subseteq S} \frac{(t-1)!(s-t)!}{s!} \cdot C(T) \\ &\leq \sum_{t=1}^{n} \binom{s}{t} \cdot \frac{(t-1)!(s-t)!}{s!} \cdot C(S) \\ &= H_n \cdot C(S)\,, \end{aligned}$$

where the inequality is because C is non-decreasing. □

Theorem: Given subadditive costs, any (1-BB w.r.t. costs C) sequential stand-alone mechanism is n-EFF.

Proof: Let S be the set chosen by the sequential stand-alone mechanism, and T be a socially optimal set. Denote by $\boldsymbol{s}, \boldsymbol{t} \in \{0,1\}^n$ the corresponding allocation vectors. For each $i \in [n]$, define

$$\sigma_i := C(S \cap [i]) - C(S \cap [i-1]) + (1 - s_i) \cdot v_i\,.$$

Then $\sum_{i=1}^n \sigma_i = C(S) + \sum_{i \notin S} v_i = SC(S)$. Consequently, it suffices to show that each $\sigma_i \leq SC(P)$.

By the definition of sequential stand-alone mechanisms, if $i \in S$, then $\sigma_i = C(S \cap [i]) - C(S \cap [i-1]) \leq v_i$ due to the definition of sequential stand-alone mechanisms

and $\sigma_i \leq C(\{i\})$ due to subadditivity. If $i \notin S$, then $\sigma_i = v_i < C((S \cap [i-1]) \cup \{i\}) - C(S \cap [i-1]) \leq C(\{i\})$ due to the definition of sequential stand-alone mechanisms and due to subadditivity.

Now, if $i \in P$ then $C(\{i\}) \leq C(P) \leq SC(P)$. Otherwise, if $i \notin P$, then $v_i \leq \sum_{j \notin P} v_i \leq SC(P)$. Consequently, $\sigma_i \leq SC(P)$. □

Note: This bound is attained if $C(S) = 1$ for all $S \neq \emptyset$ and $v_i = 1 - \varepsilon$ for all players $i \in [n]$.

2.9. Characterizing Collusion-Resistant Cost Sharing

Lemma: (Moulin, 1999) Suppose ξ is the cost-sharing method induced by a 2-GSP cost-sharing mechanism. Then for all $A \subseteq [n]$ and all $j, k \in [n]$ at least one of the following conditions holds:

i) $\xi_j(A \cup \{j,k\}) < \xi_j(A \cup \{j\})$ and $\xi_k(A \cup \{j,k\}) < \xi_k(A \cup \{k\})$

ii) $\xi_j(A \cup \{j,k\}) = \xi_j(A \cup \{j\})$

iii) $\xi_k(A \cup \{j,k\}) = \xi_k(A \cup \{k\})$

Theorem: WSGSP $\Rightarrow$ separable

Proof: (by induction over $m \in [n]_0$) Let $\boldsymbol{v}$ be the true valuations. Define $\boldsymbol{b}$ by $b_i = b^\infty$ if $i \in Q(\boldsymbol{v})$ and $b_i = -1$ otherwise. Define also $\boldsymbol{b}^S := (\boldsymbol{v}_{-S}, \boldsymbol{b}_S)$.

IH: $\forall S \subseteq [n], |S| \leq m : M_S(\boldsymbol{v}) = M_S(\boldsymbol{b}_S)$ and $u(\boldsymbol{v}) = u(\boldsymbol{b}_S)$

IS ($m-1 \to m$): Suppose $S = T \cup i$, $|T| \leq m-1$

i) $u_S(\boldsymbol{b}^S) \leq u_S(\boldsymbol{v})$, otherwise some $i \in S$ could improve at $\boldsymbol{b}^{S \setminus i}$ by bidding b_i.

ii) $u_S(\boldsymbol{b}^S) \geq u_S(\boldsymbol{v})$. Otherwise, since $q_S(\boldsymbol{v}) = q_S(\boldsymbol{b}^S)$ and due to (i), S could help some $i \in S$ at $\boldsymbol{b}^S$ by bidding $\boldsymbol{v}_S$.

Consequently, $M_S(\boldsymbol{v}) = M_S(\boldsymbol{b}^S)$, and GSP implies $u(\boldsymbol{v}) = u(\boldsymbol{b}^S)$. □

Theorem: (Moulin, 1999) GSP + 1-BB w.r.t. submodular costs $\Rightarrow$ cross-monotonic cost shares

Proof: Suppose ξ is not cross-monotonic. Let $A \subset [n]$ be so that $|A|$ is minimal and there are $i, k \notin A$, $i \neq k$, with $\xi_k(A \cup k) < \xi_k(A \cup \{i,k\})$.

Observe that

- $\xi_A(A) \geq \xi_A(A \cup j)$ due to minimality of $|A|$,
- $\xi_j(A \cup j) = \xi_j(A \cup \{j,k\})$ due the lemma above,
- and $\xi_{A \cup k}(A \cup k) \leq \xi_{A \cup k}(A \cup \{j,k\})$ due to semi-cross monotonicity.

Hence,

$$\begin{aligned} C(A \cup j) - C(A) &= \sum_{i \in A \cup j} \xi_i(A \cup j) - \sum_{i \in A} \underbrace{\xi_i(A)}_{\geq \xi_i(A \cup j)} \\ &\leq \xi_j(A \cup j) = \xi_j(A \cup \{j,k\}) \\ &= C(A \cup \{j,k\}) - \sum_{i \in A \cup i} \underbrace{\xi_i(A \cup \{j,k\})}_{\geq \xi_i(A \cup k)} \\ &= C(A \cup \{j,k\}) - C(A \cup k) \end{aligned}$$

Theorem: (Immorlica et al., 2008) GSP + upper-continuous $\Rightarrow$ cross-monotonic cost shares

Proof: By way of contradiction: Let $S \subseteq [n]$, $j \in S$, and $k \notin S$ with so that $\xi_j(S \cup k) > \xi_j(S)$.

Let $\boldsymbol{v}$ be the true valuation vector with $v_i = b^\infty$ for all $i \in S \setminus k$, $v_k = \xi_k(S \cup k)$, and $v_i = -1$ for all $i \in [n] \setminus (S \cup k)$. By upper-continuity, we have $k \in Q(\boldsymbol{v})$. Now, player k could help j at $\boldsymbol{v}$ by bidding -1.

Theorem: (Schummer, 2000) Bribe-proof $\Rightarrow$ every $\theta_i(\cdot)$ is a constant

Proof: Let $\boldsymbol{v}$ be the true valuation vector, $i \in [n]$, and $j \in [n] \setminus i$. We show that

i) v_i is a local maximizer of $b_i \mapsto u_j(\boldsymbol{v}_{-i}, b_i)$,

ii) $b_i \mapsto u_j(\boldsymbol{v}_{-i}, b_i)$ is continuous, and

iii) any continuous function $f : \mathbb{R} \to \mathbb{R}$ where every $x \in \mathbb{R}$ is a local maximizer is a constant.

(i) Due to the threshold property, there is $\delta > 0$ so that for all $b_i \in \mathbb{R}$ with $|v_i - b_i| < \delta$ it holds that $u_i(\boldsymbol{v}_{-i}, b_i) = u_i(\boldsymbol{v})$. Consequently, bribe-proofness implies $u_j(\boldsymbol{v}_{-i}, b_i) \leq u_j(\boldsymbol{v})$ for every such b_i.

(ii) By way of contradiction, suppose $b_i \mapsto u_j(\boldsymbol{v}_{-i}, b_i)$ is not continuous at v_i. Then (also due to (i)) there are $0 < \delta < \varepsilon$ and an i-variant $\boldsymbol{b}$ of $\boldsymbol{v}$ with $|b_i - v_i| < \delta$, $u_i(\boldsymbol{v}_{-i}, b_i) = u_i(\boldsymbol{v})$, and $u_j(\boldsymbol{v}_{-i}, b_i) + \varepsilon \leq u_j(\boldsymbol{v})$. Now, player j can bribe i at $(\boldsymbol{v}_{-i}, b_i)$ by paying δ because $u_i(\boldsymbol{v}_{-i}, b_i \mid b_i) - u_i(\boldsymbol{v} \mid b_i) \leq |b_i - v_i| < \delta < \varepsilon$. A contradiction.

(iii) By way of contradiction, let $x, y \in \mathbb{R}$ so that w.l.o.g. $x < y$ and $f(x) < f(y)$. Since f is continuous, $z := \max\left(f^{-1}(\{x\}) \cap [x, y]\right)$ is well-defined. However, due to the intermediate value theorem, z is not a local maximizer. □

Theorem: Dutta-Ray mechanisms with most-cost-efficient set and price selection are $2H_n$-EFF.

Proof: Let Q be the set output by the acyclic mechanism for $\boldsymbol{v}$, and P be a set with optimum social cost.

$$\begin{aligned} SC(Q) = C'(Q) + \sum_{i \notin Q} v_i &\leq \sum_{i \in Q \cap P} x_i + \sum_{i \in Q \setminus P} x_i + \sum_{i \notin Q} v_i \\ &\leq \sum_{i \in Q \cap P} x_i + \sum_{i \in P \setminus Q} v_i + \sum_{i \notin P} v_i \,, \end{aligned}$$

hence,

$$\frac{SC(Q)}{C(P)+\sum_{i\notin P} v_i} \leq \frac{\sum_{i\in Q\cap P} x_i + \sum_{i\in P\setminus Q} v_i}{C(P)}\,.$$

We first bound $\sum_{i\in Q\cap P} x_i$. So let $i \in Q\cap P$. Suppose it was accepted in iteration k, and the remaining set of players were Q_k, the set of already accepted players N_k. Because of most-cost-efficient set selection and subadditivity, we have

$$x_i \leq \frac{C(N_k \cup (Q\cap P)) - C(N_k)}{|(Q\cap P)\setminus N_k|} \leq \frac{C(Q\cap P)}{|(Q\cap P)\setminus N_k|}\,.$$

Now the "worst-case" happens when players are dropped one-by-one. Then, $\sum_{i\in Q\cap P} x_i \leq H_{|Q\cap P|}\cdot C(Q\cap P)$. Finally, we bound $\sum_{i\in P\setminus Q} v_i$. There is a player $i\in P\setminus Q$ with $v_i < \frac{C(P\setminus Q)}{|P\setminus Q|}$ because otherwise $P\setminus Q$ would have been accepted at the time the first player of $P\setminus Q$ was rejected. Repeating the argument gives $\sum_{i\in P\setminus Q} v_i \leq H_{|P\setminus Q|}\cdot C(P\setminus Q)$. The proof follows. □

Theorem: The prices offered by Dutta-Ray mechanisms are increasing over iterations.

Proof: Since

$$\frac{C(N\cup S)-C(N)}{|S|} \leq \frac{C(N\cup S\cup S')-C(N)}{|S|+|S'|}$$

it also holds that

$$\frac{C(N\cup S)-C(N)}{|S|} \leq \frac{C(N\cup S\cup S')-C(N\cup S)}{|S'|}$$

because of $\frac{a}{b}\leq\frac{c}{d} \Leftrightarrow ad\leq cb \Leftrightarrow ad-ab\leq cb-ab \Leftrightarrow \frac{a}{b}\leq\frac{c-a}{d-b}$. □

Definition: (Brenner and Schäfer, 2008) A cost-sharing method ξ is said to be weakly monotone with respect to costs C' if for all sets of players $A\subseteq B\subseteq[n]$ it holds that $\sum_{i\in A}\xi_i(B)\geq C'(A)$.

Theorem: (Brenner and Schäfer, 2008) Let M be a sequential stand-alone mechanism so that its induced cost-sharing method ξ is weakly monotone with respect to the actual costs C'. Let the optimal costs be C, and suppose it holds for all $A,B\subseteq[n]$ that $C'(A\cup B)\leq \alpha\cdot(C(A)+C(B))$. Then, M is α-EFF.

Proof: Let Q be the set output by the sequential mechanism for $\boldsymbol{v}$, and P be a set with optimum social cost.

$$\begin{aligned}\frac{SC(Q)}{C(P)+\sum_{i\notin P} v_i} &= \frac{C'(Q)+\sum_{i\in P\setminus Q} v_i + \sum_{i\notin Q\cup P} v_i}{C(P)+\sum_{i\in Q\setminus P} v_i + \sum_{i\notin Q\cup P} v_i}\\ &\leq \frac{C'(Q)+\sum_{i\in P\setminus Q} v_i}{C(P)+\sum_{i\in Q\setminus P} v_i}\\ &\leq \frac{C'(Q)+\sum_{i\in P\setminus Q} v_i}{C(P)+C(Q\setminus P)}\end{aligned}$$

The last line is due to weak monotonicity: $\sum_{i\in Q\setminus P} v_i \geq \sum_{i\in Q\setminus P}\xi_i(Q)\geq C'(Q\setminus P)\geq C(Q\setminus P)$. It remains to be shown that $\sum_{i\in P\setminus Q} v_i \leq C'(P\cup Q)-C'(Q)$.

Denote by $p_1<\cdots<p_\ell$ the players in $P\setminus Q$. For each $i\in[\ell]$, let R_i be the set of remaining players in $P\cup Q$ immediately before player p_i is considered, i.e., $p_i\in R_i$. Then, $v_{p_i}<\xi_{p_i}(R_i)$ and

$$\begin{aligned}C'(R_i) &= \sum_{j\in R_i}\xi_j(R_i)\\ &= \xi_{p_i}(R_i)+\sum_{j\in R_{i+1}}\xi_j(R_i)\\ &\geq \xi_{p_i}(R_i)+C'(R_{i+1})\,,\end{aligned}$$

hence

$$\begin{aligned}\sum_{i\in P\setminus Q} v_i &< \sum_{i=1}^{\ell}\xi_{p_i}(R_i) \leq \sum_{i=1}^{\ell}(C'(R_i)-C(R_{i+1}))\\ &\leq C'(P\cup Q)-C'(Q)\,.\end{aligned}$$

Lemma: Effectively pairwise SP = 2-GSP.

Proof: Suppose $\{i,j\}$ is a successful 2-GSP coalition so that player j can further improve at $\boldsymbol{b}$. Due to the threshold property, this implies $j\notin Q(\boldsymbol{b})$ and $\theta_{-j}(\boldsymbol{b}_{-j})<v_i$. Hence, also $b_j<v_j$ and $u_j(\boldsymbol{b})=u_j(\boldsymbol{v})=0$.

Since $\{i,j\}$ is successful and player j does not improve, we have $u_i(\boldsymbol{b})>u_i(\boldsymbol{v})$. Due to SP, we also have $u_i(\boldsymbol{v}_{-j},b_j)=u_i(\boldsymbol{b})$.

Consequently, $\{i,j\}$ is also successful at $\boldsymbol{v}$ when bidding v_i and b_j, respectively. Clearly, neither i nor j could further improve afterwards. □

2.10. Randomization

Beispiel: GSP in expectation $\neq$ randomization over deterministic GSP mechanisms. Consider two 2-player mechanisms M and M': Define M by $\theta_1(\boldsymbol{b})=10$ and

$$\theta_2(\boldsymbol{b}) = \begin{cases}1 & \text{if } b_1\leq 10\\ 10 & \text{if } b_1>10.\end{cases}$$

Note that M is GSP and 1-BB w.r.t. supermodular costs, $C(\{1\})=10$, $C(\{2\})=1$, $C(\{1,2\})=20$. Similarly, let mechanisms M' equal to M with the roles of the two players swapped.

Consider now the randomized mechanism that chooses M and M' with probability $\frac{1}{2}$ each. Suppose $\boldsymbol{v}=(11,11)$. The expected utility is 1 for each player. However, $\boldsymbol{b}=(5,5)$ gives an expected utility of 5 for each player.

Note: Counter examples are possible where players increase their bids, and for two-price mechanisms: Consider ξ defined by:

1		
2	1	
2	1	1

Consider a randomization over all 3! orders. Suppose $\boldsymbol{v} = (2-\varepsilon, 2-\varepsilon, 2-\varepsilon)$. Then the expected utilities are $u(\boldsymbol{v}) = (\frac{1-\varepsilon}{3}, \frac{1-\varepsilon}{3}, \frac{1-\varepsilon}{3})$. However, with $\boldsymbol{b} = (2,2,2)$, we have $u(\boldsymbol{b}) = (\frac{2}{3}-\varepsilon, \frac{2}{3}-\varepsilon, \frac{2}{3}-\varepsilon)$.

2.11. General-Demand Cost Sharing

WGSP general-demand mechanisms were given by:

- Moulin (1999): Incremental mechanisms for supermodular costs and increasing marginal costs
- Devanur et al. (2005): Mechanisms for multicover, a generalization of the set cover cost-sharing problem,
- Mehta et al. (2007): Acyclic mechanisms

Note: In general, incremental mechanisms w.r.t. supermodular are not GSP:

S	1	2	3	1,2	1,3	2,3	1,2,3
$C(S)$	1	1	1	3	2	3	5

and $\boldsymbol{v} = (1, 1.5, 1.5)$. Then $Q(\boldsymbol{v}) = \{2\}$ and $x(\boldsymbol{v}) = (0,2,0)$. Now for $\boldsymbol{b} = (1.5, 1.5, 1.5)$, we have $Q(\boldsymbol{b}) = \{1,3\}$ and $x(\boldsymbol{b}) = (1,0,1)$.

3. Outside the Realm of Cost Sharing

3.1. "Incentive-compatible" approximation algorithms

Scheduling on unrelated machines: The type of each machine $i \in [n]$ is specified by a vector of processing times t_{ij} for each job $j \in [m]$. An alternative is an allocation $\boldsymbol{a} \in \{0,1\}^{n\times m}$. We have $t_i(\boldsymbol{a}) = -\sum_{i\in[m]} t_{ij}a_{ij} = -\langle \boldsymbol{t}_i, \boldsymbol{a}\rangle$.

Suppose $\boldsymbol{t}, \boldsymbol{t}'$ are i-variants and let $\boldsymbol{a} := q(\boldsymbol{t}) \neq q(\boldsymbol{t}') =: \boldsymbol{b}$. Weak monotonicity implies $\langle \boldsymbol{t}_i - \boldsymbol{t}'_i, \boldsymbol{a} - \boldsymbol{b}\rangle \leq 0$.

3.2. k-Strong Price of Anarchy

Andelman et al. (2007) define a k-strong equilibrium as a state where no coalition of up to k player can strictly improve the utility of *all* members by a pure deviation. They scheduling of selfish jobs on identical and unrelated machines.

Theorem: In a job scheduling game, a strategy profile with lexicographically minimal (sorted) load vector is a k-strong equilibrium.

Note: This theorem (i.e., existence of strong equilibria) crucially relies on that all players strictly improve. For example, consider the following setting: 2 identical machines, 3 identical jobs. Suppose two jobs share one machine, and the third job is the the other machine. Now pair of jobs on the same machine can improve when only one switches to the other machine.

References

N. Andelman, M. Feldman, and Y. Mansour. Strong price of anarchy. In *Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'07)*, pages 189–198. Society for Industrial and Applied Mathematics, 2007. 9

K. J. Arrow. *Social Choice and Individual Values*. John Wiley & Sons, 2nd edition, 1963. 3

J. Brenner and G. Schäfer. Singleton acyclic mechanisms and their applications to scheduling problems. In Monien and Schroeder (2008), pages 315–326. DOI: 10.1007/978-3-540-79309-0_28. 8

R. Deb and L. Razzolini. Voluntary cost sharing for an excludable public project. *Mathematical Social Sciences*, 37(2):123–138, 1999. DOI: 10.1016/S0165-4896(98)00026-2. 1

N. R. Devanur, M. Mihail, and V. V. Vazirani. Strategyproof cost-sharing mechanisms for set cover and facility location games. *Decision Support Systems*, 39(1):11–22, 2005. DOI: 10.1016/j.dss.2004.08.004. 5, 9

P. Dhangwatnotai, S. Dobzinski, S. Dughmi, and T. Roughgarden. Truthful approximation schemes for single-parameter agents. In *Proceedings of the 49th Annual Symposium on Foundations of Computer Science (FOCS'08)*, pages 15–24, 2008. DOI: 10.1109/FOCS.2008.71. 4

S. Dobzinski, A. Mehta, T. Roughgarden, and M. Sundararajan. Is Shapley cost sharing optimal? In Monien and Schroeder (2008), pages 327–336. DOI: 10.1007/978-3-540-79309-0_29. 6

B. Dutta and D. Ray. A concept of egalitarianism under participation constraints. *Econometrica*, 57(3):615–635, 1989. URL http://www.jstor.org/stable/1911055. 1

J. Feigenbaum, A. Krishnamurthy, R. Sami, and S. Shenker. Hardness results for multicast cost sharing. *Theoretical Computer Science*, 304(1–3):215–236, July 2003. DOI: 10.1016/S0304-3975(03)00085-9. 6

A. Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601, 1973. URL http://www.jstor.org/stable/1914083. 3

J. Green and J.-J. Laffont. Characterizations of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica*, 45(2):427–438, 1977. URL http://www.jstor.org/stable/1911219. 4

T. Groves. Incentives in teams. *Econometrica*, 41(4):617–631, 1973. URL http://www.jstor.org/stable/1914085. 4

S. Hart and A. Mas-Colell. Potential, value, and consistency. *Econometrica*, 57(3):589–614, 1989. URL http://www.jstor.org/stable/1911054. 2

N. Immorlica, M. Mahdian, and V. S. Mirrokni. Limitations of cross-monotonic cost-sharing schemes. *ACM Transactions on Algorithms*, 4(2):1–25, May 2008. DOI: 10.1145/1361192.1361201. 1, 5, 7

A. Mehta, T. Roughgarden, and M. Sundararajan. Beyond Moulin mechanisms. In *Proceedings of the 8th ACM Conference on Electronic Commerce (EC'07)*, pages 1–10, 2007. DOI: 10.1145/1250910.1250912. URL http://dx.doi.org/10.1145/1250910.1250912. Full version available at http://theory.stanford.edu/~tim/papers/bmm.pdf. 9

B. Monien and U.-P. Schroeder, editors. *Proceedings of the 1st International Symposium on Algorithmic Game Theory (SAGT'08)*, volume 4997 of *LNCS*, 2008. DOI: 10.1007/978-3-540-79309-0. 9

H. Moulin. Incremental cost sharing: Characterization by coalition strategy-proofness. *Social Choice and Welfare*, 16(2):279–320, 1999. DOI: 10.1007/s003550050145. 4, 7, 9

H. Moulin and S. Shenker. Strategyproof sharing of submodular costs: budget balance versus efficiency. *Economic Theory*, 18(3):511–533, 2001. DOI: 10.1007/PL00004200. 4, 6

S. Mutuswami. Strategyproof cost sharing of a binary good and the egalitarian solution. *Mathematical Social Sciences*, 48(3):271–280, 2004. DOI: 10.1016/j.mathsocsci.2004.03.002. 1

N. Nisan. Introduction to mechanisms design (for computer scientists). In N. Nisan, T. Roughgarden, É. Tardos, and V. V. Vazirani, editors, *Algorithmic Game Theory*, chapter 9. Cambridge University Press, 2007. 4

M. J. Osborne and A. Rubinstein. *A course in game theory*. MIT Press, 1994. 2

M. Pál and É. Tardos. Group strategyproof mechanisms via primal-dual algorithms. In *Proceedings of the 44th Annual Symposium on Foundations of Computer Science (FOCS'03)*, pages 584–593, 2003. DOI: 10.1109/SFCS.2003.1238231. 5

D. C. Parkes. *Iterative Combinatorial Auctions: Achieving Economic and Computational Efficiency*. PhD thesis, Department of Computer and Information Science, University of Pennsylvania, May 2001. URL http://www.eecs.harvard.edu/~parkes/diss.html. 4

K. Roberts. The characterization of implementable choice rules. In J.-J. Laffont, editor, *Aggregation and Revelation of Preferences*, pages 321–349. North-Holland, 1979. 3

T. Roughgarden and M. Sundararajan. New trade-offs in cost-sharing mechanisms. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing (STOC'06)*, pages 79–88, 2006. DOI: 10.1145/1132516.1132528. Extended version available at http://theory.stanford.edu/~tim/papers/trade.pdf. 5, 6

M. A. Satterthwaite. Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975. DOI: 10.1016/0022-0531(75)90050-2. 3

J. Schummer. Manipulation through bribes. *Journal of Economic Theory*, 91(2):180–198, 2000. DOI: 10.1006/jeth.1999.2618. 7

L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the theory of games, Vol. II*, Annals of Mathematics Studies, No. 28, pages 307–317. Princeton University Press, Princeton, N.J., 1953. 2

L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1(1):11–26, 12 1971. DOI: 10.1007/BF01753431. 2

S. Shenker. Some technical results on continuity, strategy-proofness, and related strategic concepts. Available at ftp://parcftp.parc.xerox.com/pub/net-research/str.ps, 1993. 3

Y. Sprumont. Population monotonic allocation schemes for cooperative games with transferable utility. *Games and Economic Behavior*, 2(4):378–394, 1990. DOI: 10.1016/0899-8256(90)90006-G. 2