A congestion game is specified by a directed graph $G = (V, E)$, latency functions $\ell_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for all $e \in E$, the number of players n, designated source and target nodes $s_i, t_i \in V$ as well as the amount of traffic d_i for each player $i \in [n]$.

A flow $\boldsymbol{f}$ is a Nash equilibrium if each player's flow $\boldsymbol{f}^i$ is a solution to the following optimization problem, where $\Delta v := 0$ for $v \in V \setminus \{s_i, t_i\}$ and $\Delta s_i := -d_i$, $\Delta t_i := d_i$:

$$\begin{aligned} \min \ & \sum_{e \in E} f_e^i \cdot \ell_e(f_e) \\ \text{s.t.} \ & \sum_{(u,v)=e \in E} f_e^i - \sum_{(v,w)=e \in E} f_e^i = \Delta v && \forall v \in V \\ & f_e^i \geq 0 && \forall e \in E \end{aligned}$$

In order to make the Karush-Kuhn-Tucker conditions explicit, we define the following functions: For each $e \in E$, let $\gamma_e^i(\boldsymbol{f}^i) := -f_e^i$. For each $v \in V$, let

$$\theta_v^i(\boldsymbol{f}^i) := \sum_{(u,v)=e \in E} f_e^i - \sum_{(v,w)=e \in E} f_e^i - \Delta v \, .$$

The above problem can then be rewritten as

$$\begin{aligned} \min \ & C^i(\boldsymbol{f}^i | \boldsymbol{f}^{-i}) \\ \text{s.t.} \ & \theta_v^i(\boldsymbol{f}^i) = 0 && \forall v \in V \\ & \gamma_e^i(\boldsymbol{f}^i) \leq 0 && \forall e \in E \end{aligned}$$

Since Slater's condition is trivially fulfilled (i.e., there exists a feasible solution for which all inequality constraints are strictly satisfied), the Karush-Kuhn-Tucker conditions are necessary and sufficient. In particular: Suppose $\boldsymbol{f}$ is a Nash flow. Then for each player $i \in [n]$ there are $\boldsymbol{\alpha}^i \in \mathbb{R}^E$ and $\boldsymbol{\beta}^i \in \mathbb{R}^V$ so that:

- Primal feasibility: $\theta_v^i(\boldsymbol{f}^i) = 0$ and $\gamma_e^i(\boldsymbol{f}^i) \leq 0$
- Dual feasibility: $\alpha_e^i \geq 0$
- Complementary Slackness: $\alpha_e \cdot f_e^i = 0$
- Stationarity:

$$\nabla C^i(\boldsymbol{f}^i | \boldsymbol{f}^{-i}) + \sum_{e \in E} \alpha_e^i \cdot \nabla \gamma_e^i(\boldsymbol{f}^i) + \sum_{v \in V} \beta_v^i \cdot \nabla \theta_v^i(\boldsymbol{f}^i) = \mathbf{0} \tag{0.1}$$

Note that the partial derivatives are with respect to f_e, for $e \in E$.

In the following, we give a sufficient condition so that an atomic splittable congestion game has a unique Nash flow. Define $\ell_e^i(\boldsymbol{f}_e) := \ell_e(f_e) + f_e^i \cdot \ell_e'(f_e)$ and $\phi_e(\boldsymbol{f}_e) := (\ell_e^i(\boldsymbol{f}_e))_{i \in [n]}$.

Lemma 0.1. *Suppose for all flows* $\boldsymbol{f} \neq \tilde{\boldsymbol{f}}$ *it holds that* $\sum_{e\in E}\langle \boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e, \phi_e(\boldsymbol{f}_e) - \phi_e(\tilde{\boldsymbol{f}}_e)\rangle > 0$. *Then, the Nash equilibrium is unique.*

Proof. We start by observing that for any linear function $\kappa : \mathbb{R}^m \to \mathbb{R}$ and any two points $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ it holds that $\langle \boldsymbol{b} - \boldsymbol{a}, \nabla\kappa(\boldsymbol{a})\rangle = \kappa(\boldsymbol{b}) - \kappa(\boldsymbol{a})$.

Now by way of contradiction, assume that both $\boldsymbol{f} \neq \tilde{\boldsymbol{f}}$ are Nash equilibria. Applying the stationarity condition to $\boldsymbol{f}^i$ and and subsequently multiplying the left side of (0.1) by $(\tilde{\boldsymbol{f}}^i - \boldsymbol{f}^i)$ yields

$$\begin{aligned}
&\langle \tilde{\boldsymbol{f}}^i - \boldsymbol{f}^i, (\ell_e^i(\boldsymbol{f}_e))_{e\in E}\rangle + \sum_{e\in E}\alpha_e^i \cdot \langle \tilde{\boldsymbol{f}}^i - \boldsymbol{f}^i, \nabla\gamma_e^i(\boldsymbol{f}^i)\rangle + \sum_{v\in V}\beta_v^i \cdot \langle \tilde{\boldsymbol{f}}^i - \boldsymbol{f}^i, \nabla\theta_v^i(\boldsymbol{f}^i)\rangle \\
&= \sum_{e\in E}(\tilde{f}_e^i - f_e^i)\cdot \ell_e^i(\boldsymbol{f}_e) + \sum_{e\in E}\alpha_e^i \cdot (\gamma_e^i(\tilde{\boldsymbol{f}}^i) - \gamma_e^i(\boldsymbol{f}^i)) + \sum_{v\in V}\beta_v^i \cdot (\theta_v^i(\tilde{\boldsymbol{f}}^i) - \theta_v^i(\boldsymbol{f}^i)) \\
&= \sum_{e\in E}(\tilde{f}_e^i - f_e^i)\cdot \ell_e^i(\boldsymbol{f}_e) - \alpha_e^i \cdot \tilde{f}_e^i\,.
\end{aligned}$$

Here, the first equality is due to the above observation and the second equality follows immediately from primary feasibility and complimentary slackness. Now doing the same for $\tilde{\boldsymbol{f}}$ and summing over all $i \in [n]$, we get:

$$\begin{aligned}
&\sum_{i\in[n]}\sum_{e\in E}(\tilde{f}_e^i - f_e^i)\cdot(\ell_e^i(\boldsymbol{f}_e) - \ell_e^i(\tilde{\boldsymbol{f}}_e)) - \alpha_e^i \cdot \tilde{f}_e^i - \tilde{\alpha}_e^i \cdot f_e^i \\
&\leq \sum_{e\in E}\sum_{i\in[n]}(\tilde{f}_e^i - f_e^i)\cdot(\ell_e^i(\boldsymbol{f}_e) - \ell_e^i(\tilde{\boldsymbol{f}}_e)) \\
&= -\sum_{e\in E}\langle \boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e, \phi_e(\boldsymbol{f}_e) - \phi_e(\tilde{\boldsymbol{f}}_e)\rangle < 0\,,
\end{aligned}$$

where the first inequality follows from primary and dual feasibility, and the second inequality follows from plugging in the assumption. This is in contradiction to the the fact that we have only summed over zero-terms (0.1). ☐

In the following, we give another, slightly stronger, sufficient condition. Here, we let $D\phi(\boldsymbol{x})$ denote the derivative (i.e., Jacobian) of the mapping ϕ at $\boldsymbol{x}$.

Lemma 0.2. *Suppose that for all* $e \in E$*, the Jacobian* $D\phi_e(\boldsymbol{f}_e)$ *is positive definite. Then, the Nash equilibrium is unique.*

Proof. Let $\boldsymbol{f} \neq \tilde{\boldsymbol{f}}$ be two feasible flows. Define the path $\omega(z) := z\boldsymbol{f} + (1-z)\tilde{\boldsymbol{f}}$, where $z \in [0,1]$. By definition of the derivative and applying the chain rule,

$$\frac{d\phi_e(\omega_e(z))}{dz} = D\phi_e(\omega_e(z)) \cdot \frac{d\omega_e(z)}{dz} = D\phi_e(\omega_e(z)) \cdot (\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)\,.$$

Integrating the left and the right sides gives

$$\phi_e(\boldsymbol{f}_e) - \phi_e(\tilde{\boldsymbol{f}}_e) = \int_0^1 D\phi_e(\omega_e(z)) \cdot (\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)\, dz\,.$$

Finally, multiplying with $(\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)^T$ yields

$$\begin{aligned}\langle \boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e, \phi_e(\boldsymbol{f}_e) - \phi_e(\tilde{\boldsymbol{f}}_e)\rangle &= \int_0^1 (\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)^T D\phi_e(\omega_e(z)) \cdot (\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)\, dz \\ &= \frac{1}{2} \cdot \int_0^1 (\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)^T (D\phi_e(\omega_e(z)) + D\phi_e(\omega_e(z))^T) \cdot (\boldsymbol{f}_e - \tilde{\boldsymbol{f}}_e)\, dz \\ &> 0\,. \end{aligned}$$

□

Note that

$$D\phi_e(\boldsymbol{f}_e) = \begin{pmatrix} \partial_1 \ell_e^1(\boldsymbol{f}_e) & \dots & \partial_n \ell_e^1(\boldsymbol{f}_e) \\ \vdots & & \vdots \\ \partial_1 \ell_e^n(\boldsymbol{f}_e) & \dots & \partial_n \ell_e^n(\boldsymbol{f}_e) \end{pmatrix} = \big((1+\delta_{ij}) \cdot \ell_e'(f_e) + f_e^i \cdot \ell_e''(f_e)\big)_{i,j\in[n]}$$

where δ denotes the Kronecker delta. For instance, when all ℓ_e are affine functions $f_e \mapsto a_e f_e + b_e$, then

$$D\phi_e(\boldsymbol{f}_e) = a_e \cdot \underbrace{\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 2 \end{pmatrix}}_{=:B}$$

Some calculation shows that B is positive definite: Let $\boldsymbol{x} \in \mathbb{R}^n \setminus \boldsymbol{0}$. Then

$$\boldsymbol{x}^T B \boldsymbol{x} = \sum_{i,j=1}^n x_i x_j b_{i,j} = \sum_{i=1}^n x_i^2 + \left(\sum_{i=1}^n x_i\right)^2 > 0\,.$$

(An alternative argument is that B is the sum of an identity matrix, which is positive definite, and a matrix with the same value in all components, which is non-negativ definite.) In the following, we prove a more general uniqueness result.

Lemma 0.3. *Let $a \in \mathbb{R}_{\geq 0}$ and $\boldsymbol{b} \in \mathbb{R}^n_{\geq 0}$ so that $a + \sum_i b_i > \sqrt{n \sum b_i^2}$. Define $B = (b_i + b_j)_{i,j\in[n]}$. Then it holds for every eigenvalue λ of B that $\lambda > -a$.*

Proof. Define $c := \sum b_i$ and $d := \sum b_i^2$. Since B is the sum of two matrices of rank 1, the rank of B is at most 2. Moreover, since B is a symmetric matrix, all eigenvalues are reals and it suffices to show that all eigenvalues are non-negative. We consider three cases:

- If the rank of B is 0, then $\boldsymbol{b} = 0$ and the only eigenvalue of B is $0 > \sqrt{nd} - a \geq -a$.
- If the rank of B is 1, then the only non-zero eigenvalue of B is the trace of B, which is exactly $2c > 0 \geq -a$.
- If the rank of B is 2, define $\mathbf{1}_n := (1, \dots, 1) \in \mathbb{R}^n$. Note that $\{\mathbf{1}_n, \boldsymbol{b}\}$ is a basis for the vector space spanned by the columns of B. Suppose $\boldsymbol{x} \neq 0$ is an eigenvector of B. Then $\boldsymbol{x} = \alpha \cdot \mathbf{1}_n + \beta \cdot \boldsymbol{b}$ for some constants α and β. Suppose the corresponding non-zero eigenvalue is λ. Then, $\lambda \cdot \boldsymbol{x} = B\boldsymbol{x}$ and $B = \boldsymbol{b} \cdot \mathbf{1}_n^T + \mathbf{1}_n \cdot \boldsymbol{b}^T$, so we have

  $$\lambda \cdot (\alpha \cdot \mathbf{1}_n + \beta \cdot \boldsymbol{b}) = \boldsymbol{b} \cdot \underbrace{\mathbf{1}_n^T \cdot \alpha \cdot \mathbf{1}_n}_{=\alpha n} + \boldsymbol{b} \cdot \underbrace{\mathbf{1}_n^T \cdot \beta \cdot \boldsymbol{b}}_{=\beta c} + \mathbf{1}_n \cdot \underbrace{\boldsymbol{b}^T \cdot \alpha \cdot \mathbf{1}_n}_{=\alpha c} + \mathbf{1}_n \cdot \underbrace{\boldsymbol{b}^T \cdot \beta \cdot \boldsymbol{b}}_{=\beta d})$$

  or, equivalently,

  $$\mathbf{1}_n(\lambda\alpha - \alpha c - \beta d) + \boldsymbol{b}(\lambda\beta - \alpha n - \beta c) = 0\,.$$

  Since $\mathbf{1}_n$ and $\boldsymbol{b}$ are linearly independent, this linear combination has to be trivial, i.e.,

  $$\alpha \cdot (c - \lambda) + \beta d = 0 \quad \text{and} \quad \alpha n + \beta(c - \lambda) = 0\,.$$

  As not both α and β can be zero at the same time, the determinant of this linear system of equations (with variables α and β) has to be zero:

  $$(c - \lambda)^2 - dn = 0 \quad \text{or, equivalently,} \quad \lambda = c \pm \sqrt{nd}\,.$$

  By assumption, we have $c - \sqrt{nd} > -a$, which completes the proof. □

Theorem 0.4. *Suppose that all edges $e \in E$ have polynomial latency functions, i.e., $\ell_e(f_e) = \sum_{k=0}^{d} a_{e,k} \cdot (f_e)^k$ where $d < \frac{3n+1}{n-1}$. Then, the Nash equilibrium is unique.*

Proof. We have

$$\begin{aligned}
D\phi_e(\boldsymbol{f}_e) &= \left((1 + \delta_{ij}) \cdot \sum_{k=1}^{d} k \cdot a_{e,k} \cdot (f_e)^{k-1} + f_e^i \cdot \sum_{k=2}^{d} k \cdot (k-1) \cdot a_{e,k} (f_e)^{k-2} \right)_{i,j \in [n]} \\
&= \sum_{k=1}^{d} k \cdot a_{e,k} \cdot (f_e)^{k-2} \left((1 + \delta_{ij}) \cdot f_e + (k-1) f_e^i \right)_{i,j \in [n]} \\
&= \sum_{k=1}^{d} k \cdot a_{e,k} \cdot (f_e)^{k-2} \underbrace{\left(\left(f_e + (k-1) \cdot f_e^i \right)_{i,j \in [n]} + f_e \cdot I_n \right)}_{=:A_k}\,.
\end{aligned}$$

Fix an arbitrary $k \in [d]$. Define $\boldsymbol{b} := (f_e + (k-1) \cdot f_e^i)_{i \in [n]}$ and $B = (b_i + b_j)_{i,j \in [n]}$. For the rest of the proof, it will be sufficient to show that

$$2f_e + \sum_i b_i = (n+k+1) \cdot f_e > \sqrt{n \sum b_i^2}\,. \tag{0.2}$$

Then, according to Lemma 0.3, it holds for all eigenvalues λ of B that $\lambda > -2f_e$. Hence, the symmetric matrix $(B + 2f_e \cdot I_n)$, which is twice the symmetric part of A_k, has only strictly positive eigenvalues and is hence positive definite. Consequently, also A_k is positive definite. Since k was chosen arbitrarily, $D\phi_e(\boldsymbol{f}_e)$ is then a weighted sum of positive definite matrices and hence positive definite itself.

Now in order to show (0.2) note that

$$\begin{aligned}
\sum_{i \in [n]} b_i^2 &= \sum_{i \in [n]} (f_e + (k-1) \cdot f_e^i)^2 \\
&= n \cdot (f_e)^2 + 2(k-1)(f_e)^2 + (k-1)^2 \sum_{i \in [n]} (f_e^i)^2 \\
&\le n \cdot (f_e)^2 + 2(k-1)(f_e)^2 + (k-1)^2 (f_e)^2 \\
&= (n + k^2 - 1)(f_e)^2\,.
\end{aligned}$$

Consequently, inequality (0.2) is fulfilled if $(n+k+1)^2 > n(n+k^2-1)$. This is equivalent to $k^2(1-n) + 2k(n+1) + 3n + 1 > 0$ or (since k is non-negative)

$$k < \frac{3n+1}{n-1}\,. \qquad \square$$

Clearly, $d \le 3$ is a sufficient condition for Theorem 0.4 to hold.