Cost Sharing

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1.1. Solution Concepts for Cooperative Games

**Definition:** (Immorlica et al., 2008) A cost-sharing method $\xi$ is in the $\beta$-core w.r.t. costs $C$ if $\xi$ is $\beta$-BB and for all $S \subseteq T \subseteq [n]$ it holds that $\sum_{i \in S} \xi_i(T) \leq \beta \cdot C(S)$. Note: Immediately by definition, any $\beta$-BB cross-monotonic cost-sharing method is in the $\beta$-core.

**Definition:** (Deb and Razzolini, 1999) A cost-sharing mechanism $M$ satisfies equal treatment if for all $i, j \in [n]$ and all $b \in \mathbb{R}^n$ it holds that $b_i = b_j$ implies $M_i(b) = M_j(b)$.

**Definition:** A social choice function $f$ is pareto optimal if for every type vector $t$ there is no outcome $o' \in \mathcal{O}$ so that $U(o') > U(f(t))$.

**Definition:** (Penna et al.) A cost-sharing mechanism is renameproof if for all players $i, j \in [n]$, all true valuations $v$ and all $(i, j)$-variants $b$ with $b_j = v_i$ and $b_i = v_j = -1$ it holds that $u_j(b | v_i) \leq u_i(v)$.

**Definition:** (Penna et al.) A cost-sharing mechanisms is reputationproof if the previous definition holds (at least) for all $j < i$.

**Definition:** Let $x, y \in \mathbb{R}_{\geq 0}^n$ be two allocations, $\sum x_i = \sum y_i$. Let $\tau : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$ sort the components of a vector. Vector $x$ Lorenz dominates $y$ iff for all $j \in [n]$ it holds that $\sum_{i=1}^j y_i \leq \sum_{i=1}^j x_i$, with at least one strict inequality.

**Definition:** (Dutta and Ray, 1989) The egalitarian solution is the set of all stable Lorenz-undominated allocations, where an allocation is stable if no (proper) subcoalition admits a better stable Lorenz-undominated allocation. (Better: At least one strictly improves, nobody becomes worse off.)

Formal definition: Given a cooperative game $v : [2^n] \rightarrow \mathbb{R}_{\geq 0}$, define the Lorenz map and Lorenz cores

$$E^n A := \{x \in A \mid \exists y \in A : \forall j \in [n] : \sum_{i=1}^j y_i \leq \sum_{i=1}^j x_i, \text{ with at least one strict inequality}\}$$

$$L(S) := \{x \in \mathbb{R}^S \mid x \text{ feasible for } S \text{ and } \exists T \subseteq S, y \in E^T L(T) : y > x_T\}.$$
Consider a cooperative game \( v : 2^N \rightarrow \mathbb{R} \), where \( N \) is a finite set of players. The Shapley value is a method for distributing the total gains contributed by the players to each player. Let \( S \subseteq [n] \) be a coalition of players. The Shapley value of \( S \) is defined as

\[
\xi_i(S) = \frac{1}{|S|!} \sum_{T \subseteq S, i \not\in T} \frac{|T|!}{|T|!} \cdot (|S| - |T| - 1)! \cdot [v(T \cup i) - v(T)].
\]

The Shapley value is unique and satisfies several desirable properties, including additivity, anonymity, and dummy player properties.

**Theorem (Sprumont, 1990):** Given that \( C \) is submodular, the Shapley value is cross-monotonic.

**Proof:** The Shapley value is the average, over all orderings of the players, of the marginal cost distributions \( \xi_i(S) = C(S \cap [i]) - C(S \cap [i - 1]) \). If \( C \) is submodular, then it is immediate that \( \xi \) is cross-monotonic.

**Example:** Consider a cooperative game where players collaborate to produce a good. The Shapley value can be used to fairly distribute the profits among the players, taking into account their marginal contributions.

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A key concept in cooperative game theory is the Shapley value, which assigns a unique payoff to each coalition. The Shapley value is defined as the average marginal contribution of a player across all possible orderings of the players. It is characterized by its properties of additivity, anonymity, and dummy player.

**Theorem:** Given a game \( v : 2^N \rightarrow \mathbb{R} \), there exists a unique potential function \( P : 2^N \rightarrow \mathbb{R} \) such that \( P(\emptyset) = 0 \) and

\[
P(S) = \sum_{T \subseteq S} \frac{(t - 1)!}{s!} \cdot (s - t) \cdot v(T).
\]

**Proof:** Denote by \( \xi : 2^N \rightarrow \mathbb{R}^n \) the Shapley-value contributions. Note that

\[
P(S \setminus i) = \sum_{T \subseteq S \setminus i} \frac{(t - 1)!}{s!} \cdot (s - t) \cdot C(T) \leq \sum_{T \subseteq S \setminus i} \frac{(t - 1)!}{s!} \cdot (s - t) \cdot C(T) = \sum_{T \subseteq S \setminus i} \frac{t!}{s!} \cdot (s - t) \cdot C(T) \leq \sum_{T \subseteq S \setminus i} \frac{t!}{s!} \cdot (s - t) \cdot C(T) = \sum_{T \subseteq S \setminus i} \frac{t!}{s!} \cdot (s - t) \cdot C(T \cup i).
\]

By the previous characterization, this completes the proof.

Note: If \( \xi \) contains the Shapley-value contributions, then for all orders \( s_1, \ldots, s_{|S|} \) of \( S \) it holds that

\[
\sum_{i=1}^{|S|} \xi_i(S) = P(S).
\]

The Shapley value is the only value with this property. The Shapley value can also be characterized in other ways. E.g., it is the only value that satisfies all of the following:

- **feasible:** \( \sum \xi_i(S) = v(S) \)
- **anonymous:**
- **additive:**
- **\( \xi_i(S) = 0 \) if marginal costs of \( i \) are always 0**

**Theorem (Sprumont, 1990):** Given that \( C \) is submodular, the Shapley value is cross-monotonic.

**Proof:** The Shapley value is the average, over all orderings of the players, of the marginal cost distributions \( \xi_i(S) = C(S \cap [i]) - C(S \cap [i - 1]) \). If \( C \) is submodular, then it is immediate that \( \xi \) is cross-monotonic.

**Shapley (1971):** proved that the core of a convex game is a polytope whose extreme points are the (usual) marginal contributions vectors. Now, a convex combination of cross-monotonic cost-sharing methods is clearly cross-monotonic, too.

**Theorem:** Let \( U \neq \emptyset \) be a finite set, \( C : 2^U \rightarrow \mathbb{R} \) be a set function. The following two statements are equivalent:

1. For all \( A, B \subseteq U \):
   - \( C(A) + C(B) \geq C(A \cup B) + C(A \cap B) \)
2. For all \( D \subseteq E \subseteq U \) and \( i \notin E \):
   - \( C(D \cup i) - C(D) \geq C(E \cup i) - C(E) \)
Proof: For the proof, we rewrite (i). For all $A, B \subseteq U : C(A) - C(A \cap B) \geq C(A \cup B) - C(B)$.

$\Rightarrow$: Let $D \subseteq E \subseteq U$ and $i \in U$. Setting $A := D \cup \{i\}$ and $B := E$ gives the desired result: $C(D \cup i) - C(D) \geq C(E) - C(E)$.

$\Leftarrow$: Let $A, B \subseteq U$ and let $a_1, \ldots, a_n$ denote the elements of $A \setminus B$ (in any arbitrary order). By assumption, we have $C((A \cap B) \cup a_1) - C(A \cap B) \geq C(B \cup a_1) - C(B)$ and similarly

$$C((A \cap B) \cup \{a_1, \ldots, a_k\}) - C((A \cap B) \cup \{a_1, \ldots, a_{k-1}\})$$

$$\geq C(B \cup \{a_1, \ldots, a_k\}) - C(B \cup \{a_1, \ldots, a_{k-1}\})$$

for all $k \in [n]$. Summing each side up for all $k \in [n]$ gives the desired result. \qed

2. Design Techniques

2.1. Mechanism Design Basics

Definition: Every player $i$ is characterized by his type $t_i \in T_i$, which determines his preference over different outcomes. That is, $U_i(o | t_i)$ is the utility of player $i$ with type $t_i$ for outcome $o \in O$.

Definition: A social choice function $f : T_1 \times \cdots \times T_n \to O$ chooses an outcome $f(t) \in O$, given types $t = (t_1, \ldots, t_n)$.

Definition: A social welfare function $F : T_1 \times \cdots \times T_n \times O \to \mathbb{R}$ ranks conceivable social states.

Definition: A mechanism $g : S_1 \times \cdots \times S_n \to O$ defines the set of strategies $S_i$ available to each player $i$, and an outcome rule such that $g(s)$ is the outcome implemented by the mechanism for strategy profile $s = (s_1, \ldots, s_n)$.

Theorem: (Revelation Principle) If there exists a mechanism $g$ that implements a social choice function $f$ in dominant strategies, then $f$ is also truthfully implementable in dominant strategies (i.e., with a strategy-proof mechanism).

2.2. Impossibility results

Definition: Player $i$’s utilities are general when for every complete and transitive ordering $\succ$ of the outcome set $O$ there is a type $t_i$ so that $U_i(\cdot | t_i)$ induces $\succ$.

Definition: neutral, unanimity, irrelevant alternatives

Theorem: (Arrow, 1963) If players have general utilities, then every social welfare function over a set of more than 2 alternatives that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

Note: A well-known special case is the Condorcet paradox. Suppose $n = 3$ and $O = \{A, B, C\}$ with $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, and $C \succ_3 A \succ_3 B$. By pairwise comparison, we get $A \succ B$, $B \succ C$, and $C \succ A$.

2.3. Restricted Domains

Definition: Suppose $O = A \times \mathbb{R}^n$, i.e., an outcome consists of an alternative $a \in A$ and monetary transfers $p \in \mathbb{R}^n$. We decompose a social choice $f(t)$ into a choice rule $q(t)$ and a payment rule $x(t)$.

Definition: A choice rule $q$ is implementable if there is a payment rule $x$ so that $(q, x)$ is implementable.

Definition: Player $i$’s utilities are quasi-linear when the type of each player $i \in [n]$ is a valuation function $t_i : A \to \mathbb{R}$ so that $U_i(a, p | t_i) = t_i(a) + p_i$. In an unpublished paper, Shenker (1993) proved several results on the relationship between various forms of truthfulness, non-bossiness, and other technical properties. However, his results do not apply in settings with quasi-linear utilities.

Definition: A domain of utility functions $U$ is monotonically closed if for all $U, V \in U$ and all $a, b \in A$ with $(1) U(a) \geq U(b) \Rightarrow V(a) \geq V(b)$ and $(2) U(a) > U(b) \Rightarrow V(A) > V(b)$ there is a utility function $W \in U$ so that for all $c \in A$ it holds that $(3) U(a) \geq U(c) \Rightarrow W(a) \geq W(c)$ and $(4) V(b) \geq V(c) \Rightarrow W(b) \geq W(c)$.

In general, the domain of quasi-linear utility functions is not monotonically closed:

A choice function $q : T \to A$ is called an affine maximizer if for some subrange $A' \subseteq A$, some player weights $w_1, \ldots, w_n \in \mathbb{R}_{>0}$ and for some outcome weights $c_a \in R$, where $a \in A'$, we have that $f(t) \in \arg \max_{a \in A'}\left(c_a + \sum_{i \in [n]} w_i \cdot t_i(a)\right)$.

Theorem: (Roberts, 1979) Suppose $|A| \geq 3$, $q : T_1 \times \cdots \times T_n \to a$ is a choice rule onto $A$, and $T_i \subseteq \mathbb{R}^A$ for all $i \in [n]$. Then $q$ is implementable in dominant strategies (= truthfully implementable due to the revelation principle) if and only if $q$ is an affine maximizer.

Definition: A choice rule $q$ satisfies weak monotonicity (WMON) if for all players $i \in [n]$, and all $t$-variants $t, t'$ with $a := q(t) \neq q(t') := b$ it holds that $t_i(a) - t_i(b) \geq t_i'(a) - t_i'(b)$.

Theorem: If a choice rule $q$ is implementable in dominant strategies then $q$ satisfies WMON. Conversely, if $f$ satisfies WMON and all $T_i \subseteq \mathbb{R}^A$ are convex sets, then $f$ is implementable in dominant strategies.
Note: It is known that WMON is not a sufficient condition for dominant strategy implementability.

**Definition:** A domain of quasi-linear utilities is called single-parameter when each valuation function is determined by a single real parameter.

Note, e.g., that much effort has been spent for devising monotone (in the machine speeds) approximation algorithms for makespan minimization on parallel related machines. A randomized PTAS is due to Dhangwatnotai et al. (2008).

### 2.4. Groves Mechanisms

**Definition:** (Groves, 1973) A cost-sharing mechanism \( M = (Q, x) \) is a Groves mechanism with respect to costs \( C \) if for all \( b \in \mathbb{R}^n \) and players \( i \in [n] \) it holds that

\[
Q(b) = \arg\max_{T \subseteq [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\}
\]

\[
x_i(b) = C(Q(b)) - \sum_{j \in Q(b) \setminus i} b_j + h_i(b_{-i}),
\]

where \( h_i : \mathbb{R}^n \setminus i \rightarrow \mathbb{R} \) is a function independent of \( b_i \).

Note: Groves mechanisms are also called VCG mechanisms (Nisan, 2007, p. 218) or Clarke-Groves mechanisms (Moulin, 1999, p. 521). The name Groves mechanisms is used, e.g., in (Parkes, 2001, p. 41).

**Theorem:** Groves mechanisms are SP.

**Proof:** Assuming \( h_i \equiv 0 \), it holds for any \( i \)-variant \( b \) of \( v \) that

\[
u_i(b) = v_i \cdot q_i(b) + \sum_{j \in Q(b) \setminus i} b_j - C(Q(b)) = \sum_{j \in Q(b)} v_j - C(Q(b)) \leq u_i(v),
\]

where the last inequality holds because \( Q(b) \) can, by definition, only be inferior to \( Q(v) \).

**Theorem:** Any cost-sharing mechanism that is both SP and 1-EFF is a Groves mechanism.

Note: For the domain with arbitrary valuation functions, a corresponding statement has been proven by Green and Laffont (1977). For the (single-parameter) cost-sharing model, the proof is easier.

**Proof:** Note first that a mechanism \( M = (Q, x) \) is a Groves mechanism if and only if for all \( i \)-variants \( b, b' \) it holds that

1. \( Q(b) \in \arg\max_{T \subseteq [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\} \)
2. When \( S := Q(b), T := Q(b') \) then \( x_i(b) - x_i(b') = \left[ C(S) - \sum_{j \in S \setminus i} b_j \right] - \left[ C(T) - \sum_{j \in T \setminus i} b_j \right] \)

By way of contradiction, suppose now that (ii) does not hold for some pair \( b, b' \). Denote by \( s, t \) the respective allocations. If \( s_i = l_i \) then \( x_i(b) = x_i(b') \) due to the threshold property. Moreover, \( C(S) = C(T) \) due to (i), so also \[ C(S) - \sum_{j \in S \setminus i} b_j = C(T) - \sum_{j \in T \setminus i} b_j \].

W.l.o.g. consider the case \( i \in S \) but \( i \notin T \). Due to the threshold property, we have

\[
\sum_{j \in S \setminus i} b_j + \theta_i(b_{-i}) - C(S) = \sum_{j \in T \setminus i} b_j - C(T),
\]

i.e., \( x_i(b) - x_i(b') = \theta_i(b_{-i}) = \left[ C(S) - \sum_{j \in S \setminus i} b_j \right] - \left[ C(T) - \sum_{j \in T \setminus i} b_j \right] \). □

**Definition:** (Moulin and Shenker, 2001) A cost-sharing mechanism is a marginal cost pricing mechanism if it is a Groves mechanism with

\[
Q(b) = \max_{T \subseteq [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\}
\]

\[
h_i(b_{-i}) = \max_{T \subseteq [n]} \left\{ \sum_{j \in T} b_j - C(T) \right\}.
\]

Here, the max around the argmax chooses the maximum subset, which is well-defined.

**Theorem:** (Moulin and Shenker, 2001) Any cost-sharing mechanism \( M \) (satisfying NPT and VP) that is both SP and 1-EFF if and only if it welfare-equivalent to a marginal cost pricing mechanism.

**Proof:** “⇒”: Due to

\[
\sum_{j \in Q(b) \setminus i} b_j - C(Q(b)) \leq h_i(b_{-i}) \leq \sum_{j \in Q(b)} b_j - C(Q(b))
\]

we have \( x_i(b) \in [0, b_i \cdot q_i(b)] \), i.e., NPT and VP hold.

⇒: From \( b_i = 0 \) we have \( h_i(b_{-i}) = \sum_{j \in Q(b) \setminus i} b_j - C(Q(b_{-i}, 0)) \), which is the same as for the marginal cost pricing mechanism.

Now suppose \( i \) is served by the marginal cost pricing mechanism but not by \( M \) (all other players receive precisely the same outcome). Then \( x_i(b) = b_i \), which proves the claim.

### 2.5. Primal-Dual Algorithms

The primal-dual schema for solving IPs:

i) Write down LP relaxation, find dual. Find some intuitive meaning for the dual variables.

ii) Start with vectors \( x = 0, y = 0 \), which are dual feasible, but primal infeasible.

iii) Until the primal is feasible

   (a) increase the dual values \( y_i \) until some dual constraint(s) go(es) tight, while maintaining dual feasibility
(b) Select some subset of the tight dual contraints, and increase the primal variable corresponding to them by an integral amount.

iv) For the analysis, prove that the output pair of vectors satisfies \( \sum c_i x_i \leq \rho \cdot \sum b_j y_j \) for as small a value of \( \rho \) as possible.

**Beispiel:** Facility Location: Input is bipartite graph \( G = (V,E) \), where \( V = F \cup N \). Each facility \( v \in F \) has opening costs \( f_v \), and each edge \( e \in E \) is associated with a connection cost \( c_e \). The triangle inequality is fulfilled. A star \( S = (f,N') \) consists of a facility \( f \) and a set of connected users \( N' \). The set of all starts is denoted by \( \mathcal{S} \). The cost \( c_S \) for a star \( S = (v,N') \) is \( c_S = f_v + \sum_{e \in N} c_e \).

Formulation as IP:

\[
\begin{align*}
\min & \quad \sum_{S \in \mathcal{S}} c_S \cdot y_S \\
\text{s.t.} & \quad \sum_{S \ni i} y_S \geq 1 \quad \text{for all } i \in C \\
& \quad y_S \in \{0,1\} \quad \text{for all } S \in \mathcal{S}
\end{align*}
\]

Dual for LP relaxation:

\[
\begin{align*}
\max & \quad \sum_{i \in N} \xi_i \\
\text{s.t.} & \quad \sum_{i \in S} \xi_i \leq c_S \quad \text{for all } S \in \mathcal{S} \\
& \quad \xi_i \geq 0 \quad \text{for all } i \in N
\end{align*}
\]

The primal-dual algorithm where dual variables are frozen once the corresponding stars are “tight” is the the basis of the SP cost-sharing mechanism by Devanur et al. (2005). Their mechanism can as well be interpreted as an egalitarian mechanism; the next set that “goes tight” is clearly the “most cost-effective” set.

A cross-monotonic cost-sharing method is obtained when cost shares correspond to the dual variables at the time they become tight; however, the dual variables continue to be increased (Pál and Tardos, 2003).

**2.6. Cost-Sharing Mechanisms**

**Acyclic mechanisms:** For every subset of players \( S \subseteq [n] \), consider the following directed graph. The vertices are the players in \( S \) and there is an edge \((i,j)\) iff \( \xi_i(S \setminus i) < \xi_j(S) \), i.e., player \( j \) benefits from removing player \( i \). This graph is acyclic. In particular, if there is an edge \((i,j)\), then the offer time \( \tau_i(S) > \tau_j(S) \).

**2.7. Lower Bounds for Cross-Monotonicity**

Many lower bounds on the approximate budget balance of families of cost-sharing mechanisms have been shown.

\( \xi \)

\( \alpha \)

\( \rho \)

\( \beta \)

\( \xi \)
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**Proof:** Let $Q$ be the set returned by the Moulin mechanism for $v$, and let $P$ be a set with optimum social cost. For the analysis, renumber the player so that $P \setminus Q = \{m + 1 \ldots n\}$, and the players in $P \setminus Q$ are deleted by the Moulin mechanism in *descending order* of their numbers. For convenience, define $P_i := P \cap [i]$. Then,

$$SC(Q) = C'(Q) + \sum_{i \notin Q} v_i \leq \sum_{i \in Q} \xi_i(Q) + \sum_{i \notin Q} v_i \tag{1}$$

$$\leq \sum_{i \in Q \cap P} \xi_i(Q) + \sum_{i \notin Q \cap P} v_i \tag{2}$$

$$< \sum_{i \in Q \cap P} \xi_i(P_1) + \sum_{i \in P \setminus Q} \xi_i(P_1) + \sum_{i \notin P} v_i \tag{3}$$

$$\leq \alpha \cdot C(P) + \sum_{i \notin P} v_i .$$

Note that (1) is due to BB and (2) is due to $\xi_i(Q) \leq v_i$ for $i \in Q$. Finally, (3) holds because $P_i \subseteq Q$ for $i \in Q \cap P$ (as $P \setminus Q = \{m + 1 \ldots n\}$) and because when $i \in P \setminus Q$ was dropped, all players in $P_i$ were still in the game. □

### 2.8. General Lower Bounds

**Theorem:** (Feigenbaum et al., 2003) In general, no SP cost-sharing mechanism can guarantee both approximate budget-balance and an approximation of the social welfare.

**Proof:** Consider the excludable public case, $C(S) = 1$ for $S \neq \emptyset$. Let $v_i = \frac{1}{n^\epsilon}$. Then, $Q(v) = [n]$ because otherwise the social welfare is 0 whereas $\sum_{i \in [n]} v_i = 1 = \frac{1}{n^\epsilon}$. Now, for every $\varepsilon > 0$ we have $i \in Q(v_{-i}, \varepsilon)$ due to the approximation guarantee of the social welfare. Consequently, $x_i(v) = \theta_i(v_{-i}) = 0$, and the mechanism is not approximate budget-balanced. □

**Theorem:** (Dobzinski et al., 2008) Any SP and $\beta$-BB cost-sharing mechanism $M = (Q, x)$ is $\Omega(\log n)$-EFF.

**Proof:** Probabilistic method: There is a “bad” instance. Let $\lambda \geq 1$ be a parameter and suppose the true valuations of each player are i.i.d. random variables with $Pr[v_i = 0] = \frac{1}{n}$ and for $1 \leq t \leq n$:

$$Pr \left[ v_i \leq \frac{t}{\lambda n} \right] = \frac{1}{n} + \int_1^{\frac{t}{\lambda n}} \frac{1}{2} \cdot 2z \, dz = \frac{1}{n} + 1 - \frac{1}{t}$$

Therefore,

$$E[v_i] = \int_{1/\lambda^n}^{1/\lambda} z \cdot \frac{1}{\lambda n z^2} \, dz = \frac{1}{\lambda n} \cdot \ln n \big|_{1/\lambda^n}^{1/\lambda} = -\frac{\ln n + \ln(\lambda n)}{\lambda n} = \ln n \frac{\ln n}{\lambda n}$$

and

$$E[v_i^2] = \int_{1/\lambda^n}^{1/\lambda} z^2 \cdot \frac{1}{\lambda n z^2} \, dz = \frac{1}{\lambda n} \left( 1 - \frac{1}{\lambda n} \right)$$

Define $V = \sum v_i$. Then, $E[V] = \frac{n \ln n}{\lambda}$ and $Var(V) = \sum Var(v_i) \leq n \cdot E[v_i^2] = \frac{1}{\lambda n^2}$. Hence, Chebyshev’s inequality yields for $t > 0$ that

$$Pr \left[ \frac{\ln n}{\lambda} - t \leq V \leq \frac{\ln n}{\lambda} + t \right] \geq 1 - Pr \left[ |V - \frac{\ln n}{\lambda}| \geq t \right] = 1 - Pr \left[ \left| V - \frac{\ln n}{\lambda} \right| \geq t \right] \geq 1 - \frac{\text{Var}[V]}{t^2} = 1 - \frac{1}{\lambda^2 t^2}.$$
2.9. Characterizing Collusion-Resistant Cost Sharing

Lemma: (Moulin, 1999) Suppose $\xi$ is the cost-sharing method induced by a 2-GSP cost-sharing mechanism. Then for all $A \subseteq [n]$ and all $j, k \in [n]$ at least one of the following conditions holds:

i) $\xi_j(A \cup \{j, k\}) < \xi_j(A \cup \{j\})$ and $\xi_k(A \cup \{j, k\}) < \xi_k(A \cup \{k\})$

ii) $\xi_j(A \cup \{j, k\}) = \xi_j(A \cup \{j\})$

iii) $\xi_k(A \cup \{j, k\}) = \xi_k(A \cup \{k\})$

Theorem: WSGSP $\Rightarrow$ separable

Proof: (by induction over $m \in [n]_0$) Let $v$ be the true valuations. Define $b$ by $b_i = b^\infty$ if $i \in Q(v)$ and $b_i = -1$ otherwise. Define also $b^S := (v_{-S}, b_S)$. Hence, $\forall S \subseteq [n], |S| \leq m : M_S(v) = M_S(b_S)$ and $u(v) = u(b_S)$.

IS ($m - 1 \rightarrow m$): Suppose $S = T \cup i$, $|T| \leq m - 1$

i) $u_S(b^S) \leq u_S(v)$, otherwise some $i \in S$ could improve at $b^S$ by bidding $b_i$.

ii) $u_S(b^S) \geq u_S(v)$. Otherwise, since $q_S(v) = q_S(b^S)$ and due to (i), $S$ could help some $i \in S$ at $b^S$ by bidding $v_S$.

Consequently, $M_S(v) = M_S(b^S)$, and GSP implies $u(v) = u(b^S)$.

Theorem: (Moulin, 1999) GSP + 1-BB w.r.t. submodular costs $\Rightarrow$ cross-monotonic cost shares

Proof: Suppose $\xi$ is not cross-monotonic. Let $A \subset [n]$ be so that $|A|$ is minimal and there are $i, k \notin A, i \neq k$, with $\xi_k(A \cup k) < \xi_k(A \cup \{i, k\})$.

Observe that

- $\xi_A(A) \geq \xi_A(A \cup j)$ due to minimality of $|A|$,
- $\xi_j(A \cup j) = \xi_j(A \cup \{j, k\})$ due to the lemma above,
- and $\xi_{A \cup k}(A \cup \{j, k\}) \leq \xi_{A \cup k}(A \cup \{j, k\})$ due to semi-cross monotonicity.

Hence,

$$C(A \cup j) - C(A) = \sum_{i \in A \cup j} \xi_i(A \cup j) - \sum_{i \in A \geq \xi_i(A \cup j)} \xi_i(A) \leq \xi_j(A \cup j) = \xi_j(A \cup \{j, k\})$$

$$= C(A \cup \{j, k\}) - \sum_{i \in A \cup k} \xi_i(A \cup \{j, k\}) \geq \xi_k(A \cup k)$$

$$= C(A \cup \{j, k\}) - C(A \cup k)$$

Theorem: (Immonativa et al., 2008) GSP + upper-continuous $\Rightarrow$ cross-monotonic cost shares

Proof: By way of contradiction: Let $S \subseteq [n], j \in S$, and $k \notin S$ with so that $\xi_j(S \cup k) > \xi_j(S)$.

Let $v$ be the true valuation vector with $v_i = b^\infty$ for all $i \in S \setminus k$, $v_k = \xi_k(S \cup k)$, and $v_i = -1$ for all $i \in [n] \setminus (S \cup k)$. By upper-continuity, we have $k \in Q(v)$. Now, player $k$ could help $j$ at $v$ by bidding $-1$.

Theorem: (Schumman, 2000) Bribe-proof $\Rightarrow$ every $\theta_i(\cdot)$ is a constant

Proof: Let $v$ be the true valuation vector, $i \in [n]$, and $j \in [n] \setminus i$. We show that

i) $v_i$ is a local maximizer of $b_i \mapsto u_j(v_{-i}, b_i)$,

ii) $b_i \mapsto u_j(v_{-i}, b_i)$ is continuous, and

iii) any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ where every $x \in \mathbb{R}$ is a local maximizer is a constant.

(i) Due to the threshold property, there is $\delta > 0$ so that for all $b_i \in \mathbb{R}$ with $|v_i - b_i| < \delta$ it holds that $u_j(v_{-i}, b_i) = u_j(v)$. Consequently, bribe-proofness implies $u_j(v_{-i}, b_i) \leq u_j(v_{-i}, b_i)$ for every such $b_i$.

(ii) By way of contradiction, suppose $b_i \mapsto u_j(v_{-i}, b_i)$ is not continuous at $v_i$. Then (also due to (i)) there are $0 < \delta < \epsilon$ and an $i$-variant $b$ of $v$ with $|b_i - v_i| < \delta$, $u_j(v_{-i}, b_i) = u_j(v)$, and $u_j(v_{-i}, b_i) + \epsilon < u_j(v)$. Now, player $j$ can bribe $i$ at $(v_{-i}, b_i)$ by paying $\delta$ because $u_i(v_{-i}, b_i \mid b_i) - u_i(v \mid b_i) \leq |b_i - v_i| < \delta < \epsilon$. A contradiction.

(iii) By way of contradiction, let $x, y \in \mathbb{R}$ so that w.l.o.g. $x < y$ and $f(x) < f(y)$. Since $f$ is continuous, $z := \max \{f^{-1}(x) \cap [x, y]\}$ is well-defined. However, due to the intermediate value theorem, $z$ is not a local maximizer.

Theorem: Dutta-Ray mechanisms with most-cost-efficient set and price selection are $2H_{\ast}$-EFF.

Proof: Let $Q$ be the set output by the acyclic mechanism for $v$, and $P$ be a set with optimum social cost.

$$SC(Q) = C'(Q) + \sum_{i \in Q} v_i \leq \sum_{i \in Q \cap P} x_i + \sum_{i \in Q \setminus P} v_i + \sum_{i \notin Q} v_i$$

$$\leq \sum_{i \in Q \cap P} x_i + \sum_{i \in P \setminus Q} v_i + \sum_{i \notin P} v_i.$$
hence,
\[
\frac{SC(Q)}{C(P) + \sum_{i \in P} v_i} \leq \sum_{i \in Q \cap P} x_i + \frac{\sum_{i \in P \cap Q} v_i}{C(P)}.
\]

We first bound \(\sum_{i \in Q \cap P} x_i\). So let \(i \in Q \cap P\). Suppose it was accepted in iteration \(k\), and the remaining set of players were \(Q_k\), the set of already accepted players \(N_k\). Because of most-cost-efficient set selection and subadditivity, we have
\[
x_i \leq \frac{C(N_k \cup (Q \cap P)) - C(N_k)}{|(Q \cap P) \setminus N_k|} \leq \frac{C(Q \cap P)}{|(Q \cap P) \setminus N_k|}.
\]

Now the “worst-case” happens when players are dropped one-by-one. Then, \(\sum_{i \in Q \cap P} x_i \leq H_{|Q \cap P|} \cdot C(Q \cap P)\).

Finally, we bound \(\sum_{i \in P \cap Q} v_i\). There is a player \(i \in P \setminus Q\) with \(v_i < \frac{C(P \setminus Q)}{|Q \setminus P|}\) because otherwise \(P \setminus Q\) would have been accepted at the time the first player of \(P \setminus Q\) was rejected. Repeating the argument gives \(\sum_{i \in P \cap Q} v_i \leq H_{|P \setminus Q|} \cdot C(P \setminus Q)\). The proof follows. \(\square\)

**Theorem:** The prices offered by Dutta-Ray mechanisms are increasing over iterations.

**Proof:** Since
\[
\frac{C(N \cup S) - C(N)}{|S|} \leq \frac{C(N \cup S \cup S') - C(N)}{|S'|}
\]
it also holds that
\[
\frac{C(N \cup S) - C(N)}{|S|} \leq \frac{C(N \cup S \cup S') - C(N \cup S)}{|S'|}
\]
because of \(\frac{a}{b} \leq \frac{c}{d} \iff ad \leq cb \iff ad - ab \leq cb - ab \iff \frac{a}{b} \leq \frac{c}{d-b}\). \(\square\)

**Definition:** (Brenner and Schäfer, 2008) A cost-sharing method \(\xi\) is said to be weakly monotone with respect to costs \(C'\) if for all sets of players \(A \subseteq B \subseteq [n]\) it holds that \(\sum_{i \in A} \xi_i(B) \geq C'(A)\).

**Theorem:** (Brenner and Schäfer, 2008) Let \(M\) be a sequential stand-alone mechanism so that its induced cost-sharing method \(\xi\) is weakly monotone with respect to the actual costs \(C'\). Let the optimal costs be \(C\), and suppose it holds for all \(A, B \subseteq [n]\) that \(C'(A \cup B) \leq \alpha \cdot (C(A) + C(B))\). Then, \(M\) is \(\alpha\)-EF.

**Proof:** Let \(Q\) be the set output by the sequential mechanism for \(v\), and \(P\) be a set with optimal social cost.
\[
\frac{SC(Q)}{C(P) + \sum_{i \in P} v_i} = \frac{C'(Q) + \sum_{i \in P \cap Q} v_i + \sum_{i \notin P \cap Q} v_i}{C(P) + \sum_{i \in Q \cap P} v_i + \sum_{i \notin Q \cap P} v_i} \leq \frac{C'(Q) + \sum_{i \in P \cap Q} v_i}{C(P) + \sum_{i \in Q \cap P} v_i} \leq \frac{C'(Q) + \sum_{i \in Q \cap P} v_i}{C'(Q) + C(P)}.
\]
The last line is due to weak monotonicity: \(\sum_{i \in Q \cap P} v_i \geq \sum_{i \in Q \cap P} \xi_i(Q) \geq C'(Q \setminus P) \geq C(Q \setminus P)\). It remains to be shown that \(\sum_{i \in P \cap Q} v_i \leq C'(P \cup Q) - C'(Q)\).

Denote by \(p_1 < \cdots < p_r\) the players in \(P \setminus Q\). For each \(i \in [r]\), let \(R_i\) be the set of remaining players in \(P \cup Q\) immediately before player \(p_i\) is considered, i.e., \(p_i \in R_i\). Then,
\[
v_{p_i} = \xi_{p_i}(R_i) \quad \text{and} \quad C'(R_i) = \sum_{j \in R_i} \xi_j(R_i) = \xi_{p_i}(R_i) + \sum_{j \in R_{i+1}} \xi_j(R_i) \geq \xi_{p_i}(R_i) + C'(R_{i+1}),
\]
hence
\[
\sum_{i \in P \setminus Q} v_{p_i} \leq \sum_{i = 1}^{r} \xi_{p_i}(R_i) \leq \sum_{i = 1}^{r} (C'(R_i) - C'(R_{i+1})) \leq C'(P \cup Q) - C'(Q).
\]

**Lemma:** Effectively pairwise SP = 2-GSP.

**Proof:** Suppose \(\{i, j\}\) is a successful 2-GSP coalition so that player \(j\) can further improve at \(b\). Due to the threshold property, this implies \(j \notin Q(b)\) and \(\theta_{-j}(b_{-j}) < v_{j}\). Hence, also \(b_j < v_j\) and \(u_j(b) = u_j(v) = 0\).

Since \(\{i, j\}\) is successful and player \(j\) does not improve, we have \(u_i(b) > u_i(v)\). Due to SP, we also have \(u_i(v_{-j}, b_j) = u_i(b)\).

Consequently, \(\{i, j\}\) is also successful at \(v\) when bidding \(v_i\) and \(b_j\), respectively. Clearly, neither \(i\) nor \(j\) could further improve afterwards. \(\square\)

2.10. Randomization

**Beispiel:** GSP in expectation ≠ randomization over deterministic GSP mechanisms. Consider two 2-player mechanisms \(M\) and \(M'\): Define \(M\) by \(\theta_1(b) = 10\) and
\[
\theta_2(b) = \begin{cases} 1 & \text{if } b_1 \leq 10 \\ 10 & \text{if } b_1 > 10. \end{cases}
\]

Note that \(M\) is GSP and 1-BB w.r.t. supermodular costs, \(C(\{1\}) = 10\), \(C(\{2\}) = 1\), \(C(\{1, 2\}) = 20\). Similarly, let mechanisms \(M'\) equal to \(M\) with the roles of the two players swapped.

Consider now the randomized mechanism that chooses \(M\) and \(M'\) with probability \(\frac{1}{2}\) each. Suppose \(v = (11, 11)\). The expected utility is 1 for each player. However, \(b = (5, 5)\) gives an expected utility of 5 for each player.

Note: Counter examples are possible where players increase their bids, and for two-price mechanisms: Consider \(\xi\) defined by:
\[
\begin{array}{ccc}
\xi_1 & 1 \\
\xi_2 & 2 \\
\xi_3 & 1 \\
\end{array}
\]
Consider a randomization over all 3! orders. Suppose $v = (2 - \varepsilon, 2 - \varepsilon, 2 - \varepsilon)$. Then the expected utilities are $u(v) = (1 + \varepsilon, 1 + \varepsilon, 1 + \varepsilon)$. However, with $b = (2, 2, 2)$, we have $u(b) = (\frac{5}{3} - \varepsilon, \frac{5}{3} - \varepsilon, \frac{5}{3} - \varepsilon)$.

### 2.11. General-Demand Cost Sharing

WGSP general-demand mechanisms were given by:

- **Moulin (1999):** Incremental mechanisms for supermodular costs and increasing marginal costs
- **Devanur et al. (2005):** Mechanisms for multicovery, a generalization of the set cover cost-sharing problem,
- **Mehta et al. (2007):** Acyclic mechanisms

Note: In general, incremental mechanisms w.r.t. supermodular are not GSP:

<table>
<thead>
<tr>
<th>$S$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1, 2, 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(S)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3, 2, 3</td>
</tr>
</tbody>
</table>

and $v = (1, 1, 5, 1, 5)$. Then $Q(v) = \{2\}$ and $x(v) = (0, 2, 0)$. Now for $b = (1, 5, 1, 5, 1, 5)$, we have $Q(b) = \{1, 3\}$ and $x(b) = (1, 0, 1)$.

### 3. Outside the Realm of Cost Sharing

#### 3.1. “Incentive-compatible” approximation algorithms

Scheduling on unrelated machines: The type of each machine $i \in [n]$ is specified by a vector of processing times $t_{ij}$ for each job $j \in [m]$. An alternative is an allocation $a \in \{0, 1\}^{n \times m}$. We have $t_i(a) = -\sum_{j \in [m]} t_{ij}a_{ij} = -\langle t_i, a \rangle$.

Suppose $t, t'$ are $i$-variables and let $a := q(t) \neq q(t') := b$. Weak monotonicity implies $\langle t_i - t_i', a - b \rangle \leq 0$.

#### 3.2. $k$-Strong Price of Anarchy

**Andelman et al. (2007)** define a $k$-strong equilibrium as a state where no coalition of up to $k$ player can strictly improve the utility of all members by a pure deviation. They scheduling of selfish jobs on identical and unrelated machines.

**Theorem:** In a job scheduling game, a strategy profile with lexicographically minimal (sorted) load vector is a $k$-strong equilibrium.

Note: This theorem (i.e., existence of strong equilibria) crucially relies on that all players strictly improve. For example, consider the following setting: 2 identical machines, 3 identical jobs. Suppose two jobs share one machine, and the third job is the the other machine. Now pair of jobs on the same machine can improve when only one switches to the other machine.

### References


Cost Sharing – Relevant Work for PhD Defense on June 15, 2009; Florian Schoppmann


